Asymptotic Aspects of the Gerber-Shiu Function in the Renewal Risk Model Using Wiener-Hopf Factorization and Convolution Equivalence

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Abstract

We study the asymptotic behavior of the Gerber-Shiu expected discounted penalty function in the renewal risk model. Under the assumption that the claim-size distribution has a convolution-equivalent density function, which allows both heavy-tailed and light-tailed cases, we establish some asymptotic formulas for the Gerber-Shiu function with a fairly general penalty function. These formulas become completely transparent in the compound Poisson risk model or for certain choices of the penalty function in the renewal risk model. A by-product of this work is an extension of the Wiener-Hopf factorization to include the times of ascending and descending ladders in the continuous-time renewal risk model.

Keywords: Asymptotics; Convolution equivalence; Duality principle; Gerber-Shiu function; Renewal risk model; Wiener-Hopf factorization

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1 Introduction

In their celebrated work, Gerber and Shiu (1998) introduced the concept of expected discounted penalty function, which is called the Gerber-Shiu function in risk theory, and they developed an elegant methodology, which is powerful in studying various actuarial problems including deriving explicit formulas for the Gerber-Shiu function in the compound Poisson risk model. Later on, Gerber and Shiu (2005) extended the study to the renewal risk model.

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Since Gerber and Shiu (1998), the number of papers published on this topic has been growing rapidly. Through these papers, the study has been greatly extended to more general risk models such as the compound Poisson risk model perturbed by diffusion, the Cox risk model, the Markov-modulated risk model, and the Lévy risk model, or to more practical situations by introducing certain economic factors such as interest rate, dividend, and stochastic return on investment. Most of these papers aim at exact calculation or integro-differential equations of the Gerber-Shiu function, and some alternatively focus on upper and lower bounds of it.

We shall restrict our attention to the ordinary renewal risk model without introducing economic factors. For this model, it is usually assumed that both claim sizes and inter-arrival times follow exponential, Erlang, or, more generally, phase-type distributions. For exact calculation or integro-differential equations, we refer to Lin and Willmot (1999, 2000), Wei and Wu (2002), Wu et al. (2003), Avram and Usábel (2004), Dickson and Drekic (2004), Li and Garrido (2004, 2005), Dickson et al. (2005), Willmot (2007), Borovkov and Dickson (2008), and Landriault and Willmot (2008), among many others. Very often it is impossible to get closed-form formulas, or the established integro-differential equations are not solvable at all, or the calculation is rather involved. For bounds, we refer to Ng and Yang (2005), Psarrakos and Politis (2008), and Psarrakos (2008).

So far, much less attention has been paid to the asymptotic behavior of the Gerber-Shiu function except the ruin probability as its special case. Among very few papers on this topic we refer to the following. Cheng and Tang (2003) derived some asymptotic formulas for the moments of the surplus prior to ruin and the deficit at ruin in the renewal risk model with convolution-equivalent claim sizes and Erlang(2) inter-arrival times. Šiaulys and Asanavičiūtė (2006) obtained an asymptotic formula for the Laplace transform of the time of ruin in the compound Poisson risk model with subexponential claim sizes. Pitts and Politis (2007) gave some qualitative discussions on the convolution equivalence of the distribution of the surplus prior to ruin.

The objective of this paper is to draw a quite complete picture for the asymptotic behavior of the Gerber-Shiu function with a fairly general penalty function in the renewal risk model with claims having a convolution-equivalent density function. We devote ourselves to deriving completely explicit asymptotic formulas for the Gerber-Shiu function. The method developed in this paper is good not only for the renewal risk model but also for many other models considered in the literature. For instance, starting from Garrido and Morales (2006) one can do the same work for the Gerber-Shiu function in the Lévy risk model.

Some people in actuarial science hold the viewpoint that asymptotic formulas are not useful based on the reasoning that these formulas are valid only when the initial capital becomes large. This is indeed a concern for poor asymptotic formulas, which could perform very badly unless the initial capital becomes extremely large. However, this is not the case for good ones. On the one hand, good asymptotic formulas usually have a simple and transparent expression which eliminates a lot of insignificant but annoying factors, and on the other hand, they are accurate enough when the initial capital becomes relatively, not extremely, large. The crucial point is how large the initial capital should be so that the asymptotic formulas make sense. In the end of this paper we shall show some numerical results to illustrate the accuracy of our asymptotic formulas, but we shall not dwell too much on this point so as to keep the paper short.

The rest of this paper consists of five sections. Precisely, after describing the model and recalling some necessary preliminaries in Section 2, we state the main result and its corollaries in Section 3. Then, after preparing a number of lemmas in Section 4, we give the proof of the main result in Section 5. Finally, we show some numerical results in Section 6.

2 The model and preliminaries

2.1 Definition of the Gerber-Shiu function

In the renewal risk model, the surplus process of the insurance company is described as

$$U_t = u + pt - \sum_{i=1}^{N_t} X_i, \qquad t \geq 0,$$

where $u \geq 0$ is the initial capital, $p > 0$ is the constant premium rate, $X_1, X_2, \dots$ denote the sizes of successive claims forming a sequence of independent, identically distributed (i.i.d.), and positive random variables with generic random variable X, common absolutely continuous distribution $F = 1 - \overline{F}$, density function f, and finite mean μ, and these claims arrive at moments $0 < \tau_1 < \tau_2 < \cdots$, which constitute a renewal counting process $N_t = \sup\{n = 0, 1, \dots : \tau_n \leq t\}$ for $t \geq 0$, with $\tau_0 = 0$. Assume that $\tau = \tau_1$ follows an absolutely continuous distribution G, density function g, and finite mean $1/\lambda$. The assumption of absolute continuity of F and G is made here just for simplicity, and we would like to point out that with a bit of extra effort one can establish similar results as in the present paper without using these assumptions. As usual, we assume the safety loading condition

$$\rho = \frac{p}{\lambda} - \mu > 0. \tag{2.1}$$

Write $S_n = \sum_{i=1}^{n} X_i$ for $n = 0, 1, \dots$, where, and throughout the paper, a sum over an empty set of indices is equal to 0 by convention.

Define the time of ruin as $T(u) = \inf\{t \geq 0 : U_t < 0 | U_0 = u\}$, where $\inf \varnothing = \infty$ by convention. Then, $U_{T(u)-}$ denotes the surplus prior to ruin and $|U_{T(u)}|$ denotes the deficit at ruin. For a constant $\delta \geq 0$ and a bivariate measurable function $\varpi(\cdot,\cdot) : [0,\infty) \times [0,\infty) \to [0,\infty)$, the well-known Gerber-Shiu expected discounted penalty function is defined as

$$\phi(u) = \mathrm{E}\left(\mathrm{e}^{-\delta T(u)} \varpi\left(U_{T(u)-}, \left|U_{T(u)}\right|\right) 1_{(T(u)<\infty)}\right), \tag{2.2}$$

where 1_A denotes the indicator of an event A. When $\delta = 0$ and $\varpi(\cdot,\cdot) \equiv 1$, the Gerber-Shiu function reduces to the ruin probability

$$\psi(u) = \Pr\left(T(u) < \infty\right).$$

2.2 Brief review of a defective renewal equation

Throughout the paper, for notational convenience we shall make some conventions. We write integrals $\int$ and $\int \cdots \int$ without showing upper/lower limits if they are taken over $(0, \infty)$ and $(0, \infty) \times \cdots \times (0, \infty)$, respectively. For two measurable functions f_1 and $f_2 : [0, \infty) \to [0, \infty)$, define

$$f_1 \star f_2(x) = \int_0^x f_1(x-y) f_2(y) \mathrm{d}y$$

and write $f^{1\star} = f$, $f^{n\star} = f \star \cdots \star f$ for every $n = 2, 3, \ldots$. For two non-decreasing functions F_1 and $F_2 : [0, \infty) \to [0, \infty)$, define

$$F_1 * F_2(x) = \int_{0-}^x F_1(x-y) F_2(\mathrm{d}y)$$

and write $F^{1*} = F$, $F^{n*} = F * \cdots * F$ for every $n = 2, 3, \ldots$. Note that the former represents the convolution of two density functions (not necessarily probabilistic ones) and the latter represents the convolution of two distribution functions (not necessarily probabilistic ones). Every time when we use the two kinds of convolution we shall let the notation speak for itself.

We follow Gerber and Shiu (1998, 2005) to derive a series expression for the Gerber-Shiu function, which is the starting point of our study. Denote by $J(x, y, t|0)$ the joint density function of $U_{T(0)-}$, $|U_{T(0)}|$, and $T(0)$. Note that, under the safety loading condition (2.1), this joint density function is defective in the sense that

$$\iiint J(x, y, t|0) \mathrm{d}x \mathrm{d}y \mathrm{d}t = \psi(0) < 1.$$

In what follows, we shall write its marginal density functions in a natural way. For instance, we write $J(x, t|0) = \int J(x, y, t|0) \mathrm{d}y$ as the joint density function of $U_{T(0)-}$ and $T(0)$, write $J(y|0) = \iint J(x, y, t|0) \mathrm{d}x \mathrm{d}t$ as the density function of $|U_{T(0)}|$, and so on. We shall use these symbols without extra explanations unless any confusion could arise.

By considering the first time the surplus falls below its initial level u, one obtains the integral equation for ϕ that

$$\phi(u) = \int_0^u \iint \mathrm{e}^{-\delta t} \phi(u-y) J(x, y, t|0) \mathrm{d}t \mathrm{d}x \mathrm{d}y + \int_u^\infty \iint \mathrm{e}^{-\delta t} \varpi(x+u, y-u) J(x, y, t|0) \mathrm{d}t \mathrm{d}x \mathrm{d}y.$$

Applying an argument of Gerber and Shiu (1998) to the renewal risk model, we see that the conditional density function of $|U_{T(0)}|$ at y, given both $U_{T(0)-} = x$ and $T(0) = t$, is equal to $f(x+y)/\overline{F}(x)$. Therefore,

$$J(x, y, t|0) = \frac{f(x+y)}{\overline{F}(x)} J(x, t|0). \tag{2.3}$$

See Gerber and Shiu (2005) or Willmot (2007) for these two formulas. Using (2.3), one can rewrite the integral equation for $\phi(u)$ as

$$\phi(u) = \int_0^u \phi(u-y) k(y) \mathrm{d}y + h(u), \tag{2.4}$$

where

$$\begin{aligned}
k(y) &= \int \frac{f(x+y)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x, \\
J_\delta(x|0) &= \int \mathrm{e}^{-\delta t} J(x,t|0)\mathrm{d}t, \\
h(u) &= \iint \varpi(x+u,y) \frac{f(x+y+u)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Note that (2.4) forms a defective renewal equation for ϕ since k is a defective density function on $(0,\infty)$,

$$\int k(y)\mathrm{d}y \le \iiint J(x,y,t|0)\mathrm{d}x\mathrm{d}y\mathrm{d}t = \psi(0) < 1.$$

Clearly, $J_0(x|0)$ coincides with $J(x|0)$, the density function of $U_{T(0)-}$. By a standard argument, the solution in form of the defective renewal equation (2.4) is given by

$$\phi(u) = \int_0^u h\left(u-y\right) v(y)\mathrm{d}y + h(u), \tag{2.5}$$

where

$$v(y) = \sum_{n=1}^{\infty} k^{n\star}(y).$$

This nice expression of the solution gives rise to the opportunity of applying the methodology based on the concept of convolution equivalence, which has appeared in various applied fields for half a century. Discussions on the asymptotic solutions of defective renewal equations can be found in, e.g. Cai and Garrido (2002) and Yin and Zhao (2006).

Introduce

$$w(x) = \int \varpi(x,y) f(x+y)\mathrm{d}y, \tag{2.6}$$

so that

$$h(u) = \int w(x+u) \frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x. \tag{2.7}$$

The function $\overline{W}(u) = \int_u^\infty w(x)\mathrm{d}x$ for $u \ge 0$ will also be used.

To be able to state our main result, we need to recall some preliminaries regarding convolution equivalence and ladder-related quantities.

2.3 Convolution-equivalent density functions

Throughout the paper, all limit relationships are for $u \to \infty$ or $x \to \infty$ unless otherwise stated, and we shall not clarify which one the underlying limit procedure is whenever the notation itself can explain it. Let a_1 and a_2 be two positive functions satisfying

$$c_* \le \liminf_{x\to\infty} \frac{a_1(x)}{a_2(x)} \le \limsup_{x\to\infty} \frac{a_1(x)}{a_2(x)} \le c^*.$$

We write $a_1 = O(a_2)$ if $c^* < \infty$, $a_1 = o(a_2)$ if $c^* = 0$, and $a_1 \asymp a_2$ if $0 < c_* \le c^* < \infty$; we write $a_1 \lesssim a_2$ if $c^* = 1$, $a_1 \gtrsim a_2$ if $c_* = 1$, and $a_1 \sim a_2$ if $c^* = c_* = 1$.

According to Chover et al. (1973a, 1973b) and Klüppelberg (1989a), a measurable function $f : [0, \infty) \to [0, \infty)$ is said to belong to the (density) class $\mathcal{L}_d(\gamma)$ for some $\gamma \ge 0$ if $f(x) > 0$ for all large $x \ge 0$ and the relation

$$\lim_{x\to\infty} \frac{f(x-y)}{f(x)} = \mathrm{e}^{\gamma y} \tag{2.8}$$

holds for all real y. Furthermore, f is said to belong to the (density) class $\mathcal{S}_d(\gamma)$ if $f \in \mathcal{L}_d(\gamma)$ and the limit

$$\lim_{x\to\infty} \frac{f^{2\star}(x)}{f(x)} = 2d \tag{2.9}$$

exists and is finite.

By Lemma 4.1(2) below, it is easy to see that if $f \in \mathcal{L}_d(\gamma)$ for some $\gamma \ge 0$ then

$$\gamma = -\lim_{x\to\infty} \frac{1}{x} \ln f(x). \tag{2.10}$$

The convergence in (2.8) is automatically uniform on compact intervals of y. Hence for every $f \in \mathcal{L}_d(\gamma)$, there is always some function $l : [0, \infty) \to [0, \infty)$ satisfying $l(x) \le x/2$, $l(x) \to \infty$, and $l(x) = o(x)$ such that

$$\lim_{x\to\infty} \sup_{0\le y\le l(x)} \left| \frac{f(x-y)}{f(x)} - \mathrm{e}^{\gamma y} \right| = 0. \tag{2.11}$$

In this paper, a measurable function $f : [0, \infty) \to [0, \infty)$ is called locally (Lebesgue) integrable if $\int_0^{x_0} f(y)\mathrm{d}y < \infty$ for every $x_0 > 0$, and f is called globally (Lebesgue) integrable if $\int_0^{\infty} f(y)\mathrm{d}y < \infty$. By Fatou's lemma, it is easy to see that every function $f \in \mathcal{S}_d(\gamma)$ is globally integrable. It is known that the constant d in (2.9) is equal to

$$\hat{f}(\gamma) = \int \mathrm{e}^{\gamma x} f(x)\mathrm{d}x;$$

see Klüppelberg (1989a) and Rogozin (2000). We shall use the notation $\hat{f}$ defined above throughout the paper. Let f_1 and $f_2 : [0, \infty) \to [0, \infty)$ be two locally integrable functions belonging to $\mathcal{L}_d(\gamma)$ for some $\gamma \ge 0$ and satisfying $f_1 \asymp f_2$. By Lemma 1.2 of Klüppelberg (1989a), $f_1 \in \mathcal{S}_d(\gamma)$ if and only if $f_2 \in \mathcal{S}_d(\gamma)$.

Let F be an absolutely continuous distribution on $[0, \infty)$ with a density function f. Clearly, if $f \in \mathcal{L}_d(\gamma)$ for some $\gamma \ge 0$ then the relation

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = \mathrm{e}^{\gamma y} \tag{2.12}$$

holds for all real y, while if $f \in \mathcal{S}_d(\gamma)$ then the relation

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\hat{f}(\gamma) \tag{2.13}$$

also holds. In the literature, relation (2.12) defines the distribution class $\mathcal{L}(\gamma)$ and the combination of relations (2.12) and (2.13) defines the distribution class $\mathcal{S}(\gamma)$. When $\gamma > 0$, a distribution F in $\mathcal{L}(\gamma)$ is usually said to have an exponential-like tail, and F in $\mathcal{S}(\gamma)$ is said to have a convolution-equivalent tail. Note that $\mathcal{S}\,(0)$ reduces to the well-known subexponential class, which contains Pareto, lognormal, and heavy-tailed Weibull distributions. The class $\mathcal{S}(\gamma)$ is often used to model claim-size distributions; see, e.g. Embrechts and Veraverbeke (1982), Klüppelberg (1989b), Tang and Tsitsiashvili (2004), Klüppelberg et al. (2004), and Doney and Kyprianou (2006).

Examples and criteria for membership of the distribution class $\mathcal{S}(\gamma)$ for $\gamma > 0$ can be found in the Theorem of Embrechts (1983) and Theorems 2-4 of Cline (1986). Most of them immediately indicate parallel examples and criteria for membership of the distribution class $\mathcal{S}_d(\gamma)$ for $\gamma > 0$. The well-known inverse Gaussian distribution has a density function

$$f(x) = \sqrt{\frac{\beta}{2\pi x^3}} \exp\left\{-\frac{\beta\,(x-\mu)^2}{2\mu^2 x}\right\}, \qquad \mu, \beta > 0, \tag{2.14}$$

which belongs to the class $\mathcal{S}_d(\gamma)$ with $\gamma = \beta/(2\mu^2)$, is bounded and eventually non-increasing; see Embrechts (1983) or Klüppelberg (1989a). The exponential density function with decay rate $\gamma > 0$ belongs to the class $\mathcal{L}_d(\gamma)$ but, unfortunately, it does not belong to the class $\mathcal{S}_d(\gamma)$.

2.4 Ladder heights and related quantities

First we introduce the following quantities for the statement of our main result, and we shall study them in detail and extend the Wiener-Hopf theory in Section 4. Denote by

$$N_+ = \inf\{n = 1, 2, \ldots : p\tau_n - S_n \geq 0\}, \qquad N_- = \inf\{n = 1, 2, \ldots : p\tau_n - S_n < 0\},$$

the numbers of innovations needed for the first (weak) ascending ladder and the first (strict) descending ladder, respectively, of the random walk $\{p\tau_n - S_n, n = 0, 1, \ldots\}$. Write

$$T_+ = \tau_{N_+}, \qquad T_- = \tau_{N_-},$$

as the times of the corresponding generic ladders, and write

$$L_+ = p\tau_{N_+} - S_{N_+}, \qquad L_- = S_{N_-} - p\tau_{N_-},$$

as the heights of the corresponding generic ladders. Note that we have defined the descending ladder height as a positive random variable. Clearly, $T(0) = T_-$. Let H_+ and H_- be the distribution functions of L_+ and L_-, respectively. Under the safety loading condition (2.1), N_+ is finite almost surely but N_- takes value ∞ with a positive probability. When $N_- = \infty$ the random variable T_- is not defined. Hence, H_+ is a proper distribution on $[0, \infty)$ while H_- is a defective distribution on $(0, \infty)$ with total mass $0 < \psi(0) < 1$.

It is worthwhile mentioning that the distribution functions of most of these quantities are transparent when the renewal risk model reduces to the compound Poisson risk model.

3 The main result and its corollaries

Recall the function w defined in (2.6). Together with $\delta \geq 0$ appearing in (2.2) and $\gamma \geq 0$ defined in (2.10), we assume that the limit

$$\alpha = -\lim_{x\to\infty} \frac{1}{x} \ln w(x)$$

exists and is nonnegative. For the three nonnegative constants δ, γ, and α, it is easy to see that we have and only have the following six mutually exclusive cases:

(1) $\delta \geq 0$, $0 \leq \alpha < \gamma$, and $\delta \vee \alpha > 0$;
(2) $\delta \geq 0$, $0 \leq \gamma < \alpha$, and $\delta \vee \gamma > 0$;
(3) $\delta \geq 0$, $0 \leq \gamma = \alpha$, and $\delta \vee \gamma > 0$;
(4) $\delta = \alpha = 0$ and $\gamma > 0$;
(5) $\delta = \gamma = 0$ and $\alpha > 0$;
(6) $\delta = \gamma = \alpha = 0$.

Actually, the first three cases are attributed to the situation that δ and $\gamma \wedge \alpha$ are not simultaneously equal to 0, while the last three make up the rest.

We shall formulate our main result into the six cases accordingly. Some assumptions concerning the regularity of f and w will be made, such as $f \in \mathcal{S}_d(\gamma)$ for some $\gamma \geq 0$ and $w \in \mathcal{L}_d(\alpha)$ for some $\alpha \geq 0$, the latter of which is easily verifiable given the exact expression of $\varpi(\cdot,\cdot)$; see (3.9) and (3.10) below. We do not intend to exhaust all possibilities of the function $\varpi(\cdot,\cdot)$, but we are keen to make our assumptions so as to allow most interesting choices of it. Note that the boundedness, integrability, and monotonicity imposed on the functions f and w are not too restrictive from a practical point of view.

Theorem 3.1. *Consider the renewal risk model introduced in Subsection 2.1. Assume that f is bounded, w is locally integrable provided $\delta \vee \alpha > 0$ or globally integrable otherwise, and* $\mathrm{E}e^{\gamma L_- - \delta T_-} < 1$.

(1) When $\delta \geq 0$, $0 \leq \alpha < \gamma$, and $\delta \vee \alpha > 0$, further assume $f \in \mathcal{S}_d(\gamma)$ and $w \in \mathcal{L}_d(\alpha)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{w(u)} = \frac{\mathrm{E}e^{-(\delta+p\alpha)\tau}}{1 - \mathrm{E}e^{\alpha X - (\delta+p\alpha)\tau}}. \tag{3.1}$$

(2) When $\delta \geq 0$, $0 \leq \gamma < \alpha$, and $\delta \vee \gamma > 0$, further assume $f \in \mathcal{S}_d(\gamma)$ and $w \in \mathcal{L}_d(\alpha)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{f(u)} = \frac{\mathrm{E}e^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}e^{\gamma X - (\delta+p\gamma)\tau}} \frac{1}{1 - \mathrm{E}e^{\gamma L_- - \delta T_-}} \iint e^{\gamma z} w(x+z) \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x\mathrm{d}z. \tag{3.2}$$

(3) When $\delta \geq 0$, $0 \leq \gamma = \alpha$, and $\delta \vee \gamma > 0$, further assume that either "$f \in \mathcal{S}_d(\gamma)$, $w \in \mathcal{L}_d(\gamma)$, $w = O(f)$" or "$f \in \mathcal{S}_d(\gamma)$, $w \in \mathcal{S}_d(\gamma)$, $f = O(w)$". Then

$$\phi(u) \sim \frac{\mathrm{E}e^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}e^{\gamma X - (\delta+p\gamma)\tau}} \left(w(u) + \frac{f(u)}{1 - \mathrm{E}e^{\gamma L_- - \delta T_-}} \iint e^{\gamma z} w(x+z) \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x\mathrm{d}z \right). \tag{3.3}$$

In particular, if $f \in \mathcal{S}_d(\gamma)$, $w \in \mathcal{S}_d(\gamma)$, and $f = o(w)$ then relation (3.3) reduces to

$$\lim_{u\to\infty} \frac{\phi(u)}{w(u)} = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1-\mathrm{E}\mathrm{e}^{\gamma X-(\delta+p\gamma)\tau}}. \tag{3.4}$$

(4) *When $\delta = \alpha = 0$ and $\gamma > 0$, further assume that $f \in \mathcal{S}_d(\gamma)$, w is eventually non-increasing such that $\overline{W}(u) < \infty$ for all large $u > 0$, and $\overline{W} \in \mathcal{L}_d(0)$. Then*

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{W}(u)} = \frac{1}{\rho}. \tag{3.5}$$

(5) *When $\delta = \gamma = 0$ and $\alpha > 0$, further assume that $\overline{F} \in \mathcal{S}_d(0)$, $w \in \mathcal{L}_d(\alpha)$, and f is eventually non-increasing. Then*

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{1}{\rho\,(1-\psi(0))} \iiint \varpi(x+z,y)J(x,y+z|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3.6}$$

(6) *When $\delta = \gamma = \alpha = 0$, further assume that either "$\overline{F} \in \mathcal{S}_d(0)$, $\overline{W} \in \mathcal{L}_d(0)$, $\overline{W} = O(\overline{F})$" or "$\overline{F} \in \mathcal{S}_d(0)$, $\overline{W} \in \mathcal{S}_d(0)$, $\overline{F} = O(\overline{W})$" and that both f and w are eventually non-increasing. Then*

$$\phi(u) \sim \frac{1}{\rho}\overline{W}(u) + \frac{\overline{F}(u)}{\rho\,(1-\psi(0))} \iiint \varpi(x+z,y)J(x,y+z|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3.7}$$

In particular, if $\overline{F} \in \mathcal{S}_d(0)$, $\overline{W} \in \mathcal{S}_d(0)$, and $\overline{F} = o(\overline{W})$ then relation (3.7) reduces to the one identical to (3.5).

We leave the proof of Theorem 3.1 to Section 5. While formulas (3.1), (3.4), and (3.5) are already explicit, the other formulas obtained in Theorem 3.1 involve the quantities

$$J_\delta(x|0), \qquad \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-}, \qquad J(x,y+z|0), \qquad \psi(0), \tag{3.8}$$

which are generally unknown in the renewal risk model. For certain special choices of the function $\varpi(\cdot,\cdot)$, it is possible to derive more transparent formulas. For this purpose, we choose

$$\varpi(x,y) = x^{r_1}y^{r_2}\mathrm{e}^{s_1x+s_2y} \tag{3.9}$$

for appropriate real numbers r_1, r_2, s_1, and s_2. This form of $\varpi(\cdot,\cdot)$ allows us to derive asymptotic formulas for the moments and moment generating functions of $U_{T(u)-}$, $|U_{T(u)}|$, and $T(u)$. If $f \in \mathcal{L}_d(\gamma)$ for some $\gamma \geq 0$, $r_2 > -1$, and $-\infty < s_2 < \gamma$, then applying the dominated convergence theorem justified by Lemma 4.1(1) below, we have

$$w(u) \sim u^{r_1}\mathrm{e}^{s_1u}f(u)\int y^{r_2}\mathrm{e}^{-(\gamma-s_2)y}\mathrm{d}y = u^{r_1}\mathrm{e}^{s_1u}f(u)\frac{\Gamma(r_2+1)}{(\gamma-s_2)^{r_2+1}}, \tag{3.10}$$

where $\Gamma(z) = \int y^{z-1}\mathrm{e}^{-y}\mathrm{d}y$ for $z > 0$ defines the gamma function. This implies $w \in \mathcal{L}_d(\alpha)$ with $\alpha = \gamma - s_1$ provided $s_1 \leq \gamma$.

Corollary 3.1. *Consider the renewal risk model introduced in Subsection 2.1. Assume that f is bounded and $\mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-} < 1$.*

(1) Choose $\varpi(x,y) = x^{r_1} y^{r_2} \mathrm{e}^{s_1 x + s_2 y}$ and assume $\delta \geq 0$, $0 < s_1 \leq \gamma$, $r_2 > -1$, $s_2 < \gamma$, $\delta \vee (\gamma - s_1) > 0$, and $f \in \mathcal{S}_d(\gamma)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{u^{r_1} \mathrm{e}^{s_1 u} f(u)} = \frac{\Gamma(r_2+1)\mathrm{E}\mathrm{e}^{-(\delta + p(\gamma - s_1))\tau}}{(\gamma - s_2)^{r_2+1}\left(1 - \mathrm{E}\mathrm{e}^{(\gamma - s_1)X - (\delta + p(\gamma - s_1))\tau}\right)}.$$

(2) Choose $\varpi(x,y) = y^{r_2}\mathrm{e}^{s_2 y}$ and assume $\delta = \gamma = 0$, $r_2 > -1$, $s_2 < 0$, $\overline{F} \in \mathcal{S}_d(0)$, and f is eventually non-increasing. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{\Gamma(r_2+1)}{\rho\,(-s_2)^{r_2+1}} + \frac{1}{\rho\,(1-\psi(0))} \int y^{r_2}\mathrm{e}^{s_2 y} \Pr\left(L_- > y\right) \mathrm{d}y. \tag{3.11}$$

In particular, if $r_2 = 0$ then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{\mathrm{E}\mathrm{e}^{s_2 L_-} - 1}{\rho s_2\,(1-\psi(0))}. \tag{3.12}$$

(3) Choose $\varpi(x,y) = \mathrm{e}^{s_2 y}$ and assume $\delta \geq 0$, $\gamma \geq 0$, $\delta \vee \gamma > 0$, $-\infty < s_2 < \gamma$, and $f \in \mathcal{S}_d(\gamma)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{f(u)} = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{(\gamma - s_2)\left(1 - \mathrm{E}\mathrm{e}^{\gamma X - (\delta+p\gamma)\tau}\right)} \frac{1 - \mathrm{E}\mathrm{e}^{s_2 L_- - \delta T_-}}{1 - \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-}}. \tag{3.13}$$

In particular, if further $s_2 = 0$ so that $\varpi(x,y) \equiv 1$, then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{\gamma X - (\delta+p\gamma)\tau}} \frac{1 - \mathrm{E}\mathrm{e}^{-\delta T_-}}{1 - \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-}}. \tag{3.14}$$

(4) Choose $\varpi(x,y) \equiv 1$ and assume $\delta > 0$, $\gamma = 0$, $f \in \mathcal{S}_d(0)$, and $\overline{F} \in \mathcal{S}_d(0)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{\mathrm{E}\mathrm{e}^{-\delta\tau}}{1 - \mathrm{E}\mathrm{e}^{-\delta\tau}}. \tag{3.15}$$

(5) Denote by $F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\mathrm{d}y$ for $x \geq 0$ the equilibrium distribution of F. Choose $\varpi(x,y) \equiv 1$ and assume $\delta = \gamma = 0$, $\overline{F} \in \mathcal{S}_d(0)$, $\overline{F_e} \in \mathcal{S}_d(0)$, and f is eventually non-increasing. Then

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F_e}(u)} = \frac{\mu}{\rho}. \tag{3.16}$$

Proof. The proof of Corollary 3.1 consists of verification and simplification of the corresponding cases of Theorem 3.1.

(1) Plug (3.10) into (3.1) with α replaced by $\gamma - s_1$.

(2) These conditions describe a special case of Theorem 3.1(6). With $r_2 = 0$, (3.12) follows from (3.11). We use (3.7) to prove (3.11). By (3.10),

$$w(u) \sim f(u)\frac{\Gamma(r_2+1)}{(-s_2)^{r_2+1}}.$$

From (2.6), it is clear that w is both eventually non-increasing and globally integrable. Furthermore,

$$\begin{aligned}\iiint \varpi(x+z,y)J(x,y+z|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint y^{r_2}\mathrm{e}^{s_2 y}J(x,y+z|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int y^{r_2}\mathrm{e}^{s_2 y}\Pr(L_- > y)\,\mathrm{d}y.\end{aligned}$$

Plugging these relations into (3.7) yields (3.11).

(3) These conditions describe a special case of Theorem 3.1(3). Using Lemma 4.4(1) below, if $\gamma > 0$ then $f(u) \sim \gamma\overline{F}(u)$. Hence, (3.14) is an immediate consequence of (3.13). We use (3.3) to prove (3.13). Note that, by (3.10), $w(u) \sim f(u)/(\gamma - s_2)$. By Lemma 4.7 below,

$$\begin{aligned}\iint \mathrm{e}^{\gamma z}w(x+z)\frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x\mathrm{d}z &= \iiint \mathrm{e}^{s_2 y+\gamma z}f(x+y+z)\frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{\mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-} - \mathrm{E}\mathrm{e}^{s_2 L_- - \delta T_-}}{\gamma - s_2}.\end{aligned}$$

Plugging these relations into (3.3) yields (3.13).

(4) Note that for this case $w = \overline{F}$. By Lemma 4.4(1), $f = o(w)$. Hence, (3.15) is a straightforward consequence of (3.4).

(5) This follows from the second assertion of Theorem 3.1(6) since $w = \overline{F}$ from the previous proof. □

Corollary 3.2(2) of Tang (2004) obtains relation (3.15) only for Pareto-like claim-size distributions but uses a totally different method. Relation (3.16) can be found in Theorem 4.6 of Embrechts and Veraverbeke (1982) and relation (3.14) with $\delta = 0$ can be found in Theorem 2(iii) of Veraverbeke (1977).

It turns out that, if we restrict our attention to the compound Poisson risk model in which the inter-arrival times follow an exponential distribution G with mean $1/\lambda$, the asymptotic formulas obtained in Theorem 3.1 are completely transparent. The key point is an identity due to Gerber and Shiu (1998), page 55: for $\delta \geq 0$,

$$J_\delta(x|0) = \frac{\lambda}{p}\mathrm{e}^{-\theta x}\overline{F}(x), \qquad (3.17)$$

where $\theta = \theta(\delta) \geq 0$ solves the Lundberg fundamental equation

$$\delta + \lambda - p\theta = \lambda \hat{f}(-\theta). \tag{3.18}$$

Clearly, if $\delta = 0$ then $\theta = 0$. Thus, for $\delta = 0$, (3.17) reduces to $J_0(x|0) = \frac{\lambda}{p}\overline{F}(x)$. Based on (3.17), we rewrite Theorem 3.1 as follows:

Corollary 3.2. *Suppose that the inter-arrival times follow an exponential distribution G with mean $1/\lambda$, so that the model introduced in Subsection 2.1 reduces to the compound Poisson risk model. Assume that f is bounded, w is locally integrable provided $\delta \vee \alpha > 0$ or globally integrable otherwise, and $\hat{f}(\gamma) < 1 + (p\gamma + \delta)/\lambda$.*

(1) When $\delta \geq 0$, $0 \leq \alpha < \gamma$, and $\delta \vee \alpha > 0$, further assume $f \in \mathcal{S}_d(\gamma)$ and $w \in \mathcal{L}_d(\alpha)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{w(u)} = \frac{\lambda}{\delta + p\alpha + \lambda - \lambda\hat{f}(\alpha)}.$$

(2) When $\delta \geq 0$, $0 \leq \gamma < \alpha$, and $\delta \vee \gamma > 0$, further assume $f \in \mathcal{S}_d(\gamma)$ and $w \in \mathcal{L}_d(\alpha)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{f(u)} = \frac{\lambda^2(\gamma+\theta)}{\left(\delta + p\gamma + \lambda - \lambda\hat{f}(\gamma)\right)^2} \iint \mathrm{e}^{-\theta x + \gamma z} w(x+z)\mathrm{d}x\mathrm{d}z.$$

(3) When $\delta \geq 0$, $0 \leq \gamma = \alpha$, and $\delta \vee \gamma > 0$, further assume that either "$f \in \mathcal{S}_d(\gamma)$, $w \in \mathcal{L}_d(\gamma)$, $w = O(f)$" or "$f \in \mathcal{S}_d(\gamma)$, $w \in \mathcal{S}_d(\gamma)$, $f = O(w)$". Then

$$\phi(u) \sim \frac{\lambda w(u)}{\delta + p\gamma + \lambda - \lambda\hat{f}(\gamma)} + \frac{\lambda^2(\gamma+\theta) f(u)}{\left(\delta + p\gamma + \lambda - \lambda\hat{f}(\gamma)\right)^2} \iint \mathrm{e}^{-\theta x + \gamma z} w(x+z)\mathrm{d}x\mathrm{d}z.$$

(4) When $\delta = \alpha = 0$ and $\gamma > 0$, further assume that $f \in \mathcal{S}_d(\gamma)$, w is eventually non-increasing such that $\overline{W}(u) < \infty$ for all large $u > 0$, and $\overline{W} \in \mathcal{L}_d(0)$. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{W}(u)} = \frac{1}{\rho}.$$

(5) When $\delta = \gamma = 0$ and $\alpha > 0$, further assume that $\overline{F} \in \mathcal{S}_d(0)$, $w \in \mathcal{L}_d(\alpha)$, and f is eventually non-increasing. Then

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{1}{\rho^2} \iiint \varpi(x+z, y) f(x+y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

(6) When $\delta = \gamma = \alpha = 0$, further assume that either "$\overline{F} \in \mathcal{S}_d(0)$, $\overline{W} \in \mathcal{L}_d(0)$, $\overline{W} = O(\overline{F})$" or "$\overline{F} \in \mathcal{S}_d(0)$, $\overline{W} \in \mathcal{S}_d(0)$, $\overline{F} = O(\overline{W})$" and that both f and w are eventually non-increasing. Then

$$\phi(u) \sim \frac{1}{\rho}\overline{W}(u) + \frac{\overline{F}(u)}{\rho^2} \iiint \varpi(x+z, y) f(x+y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Proof. Using (2.3) and (3.17), we have

$$\begin{aligned}
\mathrm{E}e^{\gamma L_- - \delta T_-} &= \iiint e^{\gamma y - \delta t} J(x,y,t|0)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \iiint e^{\gamma y - \delta t} \frac{f(x+y)}{\overline{F}(x)} J(x,t|0)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \iint e^{\gamma y} \frac{f(x+y)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y \\
&= \frac{\lambda}{p} \iint e^{-\theta x + \gamma y} f(x+y)\mathrm{d}x\mathrm{d}y. \qquad (3.19)
\end{aligned}$$

Thus, for $\delta \vee \gamma > 0$, applying (3.18) to (3.19), after some simple calculation we obtain

$$\mathrm{E}e^{\gamma L_- - \delta T_-} = \frac{\lambda \hat{f}(\gamma) - \lambda + p\theta - \delta}{p(\gamma + \theta)},$$

while for $\delta = \gamma = 0$, (3.19) directly implies $\psi(0) = \lambda\mu/p$, which is already well known. Furthermore, integrating both sides of (2.3) with respect to t then using (3.17) with $\delta = 0$, we have

$$J(x, y+z|0) = \frac{\lambda}{p} f(x+y+z).$$

So far, we have derived explicit expressions for all the unknowns listed in (3.8). Therefore, by simply plugging these expressions into the formulas obtained in Theorem 3.1 accordingly, we obtain all the claimed formulas in Corollary 3.2. □

Under the conditions of Corollary 3.2(3), for $\delta = 0$, $\gamma > 0$, and $\varpi(\cdot,\cdot) \equiv 1$ for which $w = \overline{F}$, using the fact $f(u) \sim \gamma\overline{F}(u)$ by Lemma 4.4(1) below, it is easy to see that

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}(u)} = \frac{\lambda^2 \rho\gamma}{\left(p\gamma + \lambda - \lambda\hat{f}(\gamma)\right)^2}.$$

This formula is first obtained in Theorem 6.3 of Embrechts and Veraverbeke (1982). Similarly, under the conditions of Corollary 3.2(3), with $\delta > 0$, $\gamma = 0$, and $\varpi(\cdot,\cdot) \equiv 1$, using the fact $f(u) = o\left(\overline{F}(u)\right)$ by Lemma 4.4(1) below, it is easy to see that

$$\lim_{u\to\infty} \frac{\phi(u)}{\overline{F}(u)} = \frac{\lambda}{\delta}.$$

Theorem 2 of Šiaulys and Asanavičiūtė (2006) verifies this last formula as well as (3.16), both for the compound Poisson risk model.

4 Lemmas

To prove Theorem 3.1, we need to prepare a series of important lemmas among which Lemmas 4.3, 4.5, and 4.6 are interesting on their own right.

4.1 On convolution-equivalent density functions

Lemma 4.1. *Let $f \in \mathcal{L}_d(\gamma)$ for some $\gamma \geq 0$.*

(1) For every $\varepsilon > 0$, there are some constants $c_0 > 0$ and $x_0 > 0$ such that, for all $x \geq y \geq x_0$,

$$c_0^{-1} e^{-(\gamma+\varepsilon)(x-y)} \leq \frac{f(x)}{f(y)} \leq c_0 e^{-(\gamma-\varepsilon)(x-y)}.$$

(2) For every $\varepsilon > 0$, there is some constant $c_1 > 0$ such that, for all large $x \geq 0$,

$$c_1^{-1} e^{-(\gamma+\varepsilon)x} \leq f(x) \leq c_1 e^{-(\gamma-\varepsilon)x}.$$

(3) If further f is locally integrable then $\hat{f}(\tilde{\gamma}) < \infty$ for every $0 \leq \tilde{\gamma} < \gamma$.

Proof. The second item can be proved by fixing $y = x_0$ in the first item, and the last item is a direct consequence of the second item. Hence, we only need to prove the first item. Applying Karamata's representation theorem for regularly-varying functions to the class $\mathcal{L}_d(\gamma)$, we know that $f \in \mathcal{L}_d(\gamma)$ if and only if it can be written as

$$f(x) = c(x) \exp\left\{-\int_0^x \gamma(z)\,dz\right\}, \tag{4.1}$$

where $c : [0,\infty) \to [0,\infty)$ and $\gamma : [0,\infty) \to (-\infty,\infty)$ are measurable functions such that $c(x)$ converges to a positive constant and $\gamma(x)$ converges to γ; see Bingham et al. (1989) or Klüppelberg (1989a). Hence, for every $\varepsilon > 0$ we can find some positive constants c_0 and x_0 such that, for all $x \geq z \geq y \geq x_0$,

$$c_0^{-1} \leq \frac{c(x)}{c(y)} \leq c_0 \quad \text{and} \quad \gamma - \varepsilon \leq \gamma(z) \leq \gamma + \varepsilon,$$

from which the assertion of the first item follows. □

The following lemma can be found in Klüppelberg (1989a):

Lemma 4.2. *Let $f \in \mathcal{S}_d(\gamma)$ for some $\gamma \geq 0$.*

(1) It holds for every $n = 1, 2, \ldots$ that

$$\lim_{x\to\infty} \frac{f^{n\star}(x)}{f(x)} = n\hat{f}(\gamma)^{n-1}.$$

(2) If further f is bounded, then for every $\varepsilon > 0$, there is some constant $c = c(\varepsilon) > 0$ such that, for all $n = 1, 2, \ldots$ and $x > 0$,

$$f^{n\star}(x) \leq c\left(\varepsilon + \hat{f}(\gamma)\right)^n f(x).$$

In the next lemma we give some simple results regarding convolution equivalence. They form the main tool in the proofs of the present paper. Analogues for distributions with convolution-equivalent tails are well known; see, e.g. Cline (1986) and Pakes (2004).

Lemma 4.3. *Let f_1 and $f_2 : [0,\infty) \to [0,\infty)$ be locally integrable.*

(1) If $f_1 \in \mathcal{L}_d(\gamma)$ for some $\gamma \geq 0$ and $f_2(x) = O\left(\mathrm{e}^{-\tilde{\gamma}x}\right)$ for some $\tilde{\gamma} > \gamma$ then

$$f_1 \star f_2(x) \sim f_1(x)\hat{f}_2(\gamma).$$

(2) If $f_1 \in \mathcal{S}_d(\gamma)$, $f_2 \in \mathcal{L}_d(\gamma)$ for some $\gamma \geq 0$, and $f_2 = O(f_1)$ then

$$f_1 \star f_2(x) \sim f_1(x)\hat{f}_2(\gamma) + \hat{f}_1(\gamma)f_2(x).$$

Proof. For both cases, it is easy to verify the finiteness of $\hat{f}_1$ and $\hat{f}_2$ for every occurrence since we have assumed the local integrability of f_1 and f_2.

(1) For some small $\varepsilon > 0$ such that $\gamma + \varepsilon < \tilde{\gamma}$, by Lemma 4.1(1) there are some constants $c_0 > 0$ and $x_0 > 0$ such that, uniformly for all $x \geq x - y \geq x_0$,

$$\frac{f_1(x-y)}{f_1(x)} \leq c_0 \mathrm{e}^{(\gamma+\varepsilon)y}.$$

We split $f_1 \star f_2(x)$ into two parts as

$$f_1 \star f_2(x) = \left(\int_0^{x-x_0} + \int_{x-x_0}^{x}\right) f_1(x-y)f_2(y)\mathrm{d}y. \tag{4.2}$$

For the first term, by the dominated convergence theorem justified by $\hat{f}_2(\gamma+\varepsilon) < \infty$,

$$\lim_{x\to\infty} \frac{1}{f_1(x)} \int_0^{x-x_0} f_1(x-y)f_2(y)\mathrm{d}y = \int_0^{\infty} \lim_{x\to\infty} \frac{f_1(x-y)}{f_1(x)} f_2(y)\mathrm{d}y = \hat{f}_2(\gamma).$$

For the second term, we have

$$\int_{x-x_0}^{x} f_1(x-y)f_2(y)\mathrm{d}y = O(\mathrm{e}^{-\tilde{\gamma}x}) \int_0^{x_0} f_1(y)\mathrm{d}y = o\left(f_1(x)\right),$$

where in the last step we used Lemma 4.1(2) and the local integrability of f_1. Plugging these estimates into (4.2), we obtain the desired result.

(2) Let l_1 be a function specified in relation (2.11) for f_1 and let l_2 be another function specified for f_2 in the same way. With $l = l_1 \wedge l_2$, we split $f_1 \star f_2(x)$ into three parts as

$$f_1 \star f_2(x) = \left(\int_0^{l(x)} + \int_{l(x)}^{x-l(x)} + \int_{x-l(x)}^{x}\right) f_1(x-y)f_2(y)\mathrm{d}y. \tag{4.3}$$

The first and third terms on the right-hand side of (4.3) are asymptotic to $f_1(x)\hat{f}_2(\gamma)$ and $\hat{f}_1(\gamma)f_2(x)$, respectively. For the second term, we have

$$\int_{l(x)}^{x-l(x)} f_1(x-y)f_2(y)\mathrm{d}y = O(1)\left(\int_0^{x} - \int_0^{l(x)} - \int_{x-l(x)}^{x}\right) f_1(x-y)f_1(y)\mathrm{d}y = o(1)f_1(x).$$

From these estimates we conclude the proof. □

Lemma 4.4. *Let $q : [0, \infty) \to [0, \infty)$ be a measurable function such that $\overline{Q}(x) = \int_x^\infty q(y)\mathrm{d}y < \infty$ for all large $x > 0$. For arbitrarily fixed $\gamma \geq 0$, consider the following three assertions:*

$$(A)\ q \in \mathcal{L}_d(\gamma); \qquad (B)\ \lim_{x\to\infty} \frac{q(x)}{\overline{Q}(x)} = \gamma; \qquad (C)\ \overline{Q} \in \mathcal{L}_d(\gamma).$$

(1) For every $\gamma \geq 0$, (A) implies (B) and (B) implies (C);

(2) For every $\gamma > 0$, (A) and (B) are equivalent and either (A) or (B) implies (C);

(3) If $\gamma > 0$ and q is eventually non-increasing, then (A), (B), and (C) are equivalent;

(4) If $\gamma = 0$ and q is eventually non-increasing, then (B) and (C) are equivalent but neither (B) nor (C) implies (A).

Proof. (1) First prove that (A) implies (B). When $\gamma > 0$, by the dominated convergence theorem justified by Lemma 4.1(1) we have

$$\lim_{x\to\infty} \frac{q(x)}{\overline{Q}(x)} = \lim_{x\to\infty} \left(\int_0^\infty \frac{q(x+y)}{q(x)} \mathrm{d}y \right)^{-1} = \left(\int_0^\infty \mathrm{e}^{-\gamma y} \mathrm{d}y \right)^{-1} = \gamma.$$

When $\gamma = 0$, by the local uniformity of relation (2.8) we have

$$\limsup_{x\to\infty} \frac{q(x)}{\overline{Q}(x)} \leq \limsup_{M\to\infty} \limsup_{x\to\infty} \frac{q(x)}{\int_x^{x+M} q(y)\mathrm{d}y} = \limsup_{M\to\infty} \frac{1}{M} = 0.$$

Next prove that (B) implies (C). Write $q(x)/\overline{Q}(x) = \gamma(x)$ for $x \geq x_0$, where x_0 is a positive number such that $\overline{Q}(x_0) < \infty$. Integrating both sides yields

$$\int_{x_0}^x \frac{q(y)}{\overline{Q}(y)} \mathrm{d}y = \int_{x_0}^x \gamma(y)\mathrm{d}y, \qquad x \geq x_0.$$

It follows that

$$\overline{Q}(x) = \overline{Q}(x_0) \exp\left\{ -\int_{x_0}^x \gamma(y)\,\mathrm{d}y \right\}, \qquad x \geq x_0.$$

Thus, by the representation theorem as described in (4.1) we recognize $\overline{Q} \in \mathcal{L}_d(\gamma)$.

(2) This immediately follows from (1).

(3) After being suitably truncated and normalized, q corresponds to the tail of a proper distribution supported on $[x_0, \infty)$ for some large $x_0 > 0$ and $\overline{Q}$ corresponds to the integrated tail of this proper distribution. Then, we see that the equivalence of (A), (B), and (C) is given by Lemma 3.1 of Tang (2007).

(4) Apply the same truncation and normalization as in the proof of (3). Then, we see that the equivalence of (B) and (C) is given by Theorem 3.2(1) of Su and Tang (2003). A counterexample showing that (C) does not necessarily imply (A) can be found in Example 3.1 of Su and Tang (2003). □

Lemma 4.5. *Let $q : [0,\infty) \to [0,\infty)$ be eventually non-increasing such that $\overline{Q}(x) = \int_x^\infty q(y)\mathrm{d}y < \infty$ for all large $x > 0$. Let R be a renewal measure generated by a non-lattice distribution on $[0,\infty)$ with mean $0 < m < \infty$. If $\overline{Q} \in \mathcal{L}_d(0)$ then*

$$\int_{0-}^{\infty} q(x+y)R(\mathrm{d}y) \sim \frac{1}{m}\overline{Q}(x).$$

Proof. Let x be so large that q is non-increasing on $[x,\infty)$. It follows from Blackwell's renewal theorem that

$$\lim_{x\to\infty} R\,(x, x+1] = \frac{1}{m};$$

see, e.g. page 347 of Feller (1971). That is, for every $0 < \varepsilon < 1$ there is some constant $c = c(\varepsilon) > 1$ such that, for all $z \geq c$,

$$\frac{1-\varepsilon}{m} \leq R\,(z, z+1] \leq \frac{1+\varepsilon}{m}.$$

Split the integral $\int_{0-}^{\infty} q(x+y)R(\mathrm{d}y)$ into two parts as

$$\int_{0-}^{\infty} q(x+y)R(\mathrm{d}y) = \int_{0-}^{c} q(x+y)R(\mathrm{d}y) + \sum_{n=0}^{\infty}\int_{c+n}^{c+n+1} q(x+y)R(\mathrm{d}y). \tag{4.4}$$

Clearly, $\int_{0-}^{c} q(x+y)R(\mathrm{d}y) \leq q(x)R[0,c]$. For the sum in (4.4), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\int_{c+n}^{c+n+1} q(x+y)R(\mathrm{d}y) &\leq \sum_{n=0}^{\infty} q(x+c+n)R(c+n, c+n+1]\\
&\leq \frac{1+\varepsilon}{m}\sum_{n=0}^{\infty} q(x+c+n)\\
&\leq \frac{1+\varepsilon}{m}\sum_{n=0}^{\infty}\int_{c+n-1}^{c+n} q(x+y)\mathrm{d}y\\
&= \frac{1+\varepsilon}{m}\overline{Q}(x+c-1).
\end{aligned}$$

Plugging these two estimates into (4.4) yields

$$\int_{0-}^{\infty} q(x+y)R(\mathrm{d}y) \leq q(x)R[0,c] + \frac{1+\varepsilon}{m}\overline{Q}(x+c-1). \tag{4.5}$$

Then using $\overline{Q} \in \mathcal{L}_d(0)$ and $q(x) = o\left(\overline{Q}(x)\right)$, the latter of which is due to Lemma 4.4(4), we obtain

$$\int_{0-}^{\infty} q(x+y)R(\mathrm{d}y) \lesssim \frac{1}{m}\overline{Q}(x).$$

The corresponding asymptotic lower bound can be obtained similarly. □

4.2 On the Wiener-Hopf factorization

Applying the classical Wiener-Hopf theory (see, e.g. Feller 1971) to the random walk $\{p\tau_n - S_n, n = 0, 1, \ldots\}$, we have, for every complex number Γ with $\operatorname{Re}\Gamma = 0$,

$$\left(1 - \mathrm{E}e^{-\Gamma L_+}\right)\left(1 - \mathrm{E}e^{\Gamma L_-}\right) = 1 - \mathrm{E}e^{\Gamma(X - p\tau)}. \tag{4.6}$$

Differentiating both sides of (4.6) with respect to Γ then taking $\Gamma = 0$, we easily obtain

$$\mathrm{E}L_+ \left(1 - \psi(0)\right) = \rho. \tag{4.7}$$

Next, we extend the Wiener-Hopf factorization formula (4.6) so as to include the times of ascending and descending ladders in the continuous-time renewal model.

Lemma 4.6. *For complex numbers Δ and Γ with $\operatorname{Re}\Delta = \operatorname{Re}\Gamma = 0$, it holds that*

$$\left(1 - \mathrm{E}e^{-\Gamma L_+ - \Delta T_+}\right)\left(1 - \mathrm{E}e^{\Gamma L_- - \Delta T_-}\right) = 1 - \mathrm{E}e^{\Gamma X - (\Delta + p\Gamma)\tau}. \tag{4.8}$$

Proof. According to the value of N_+, we expand

$$\mathrm{E}e^{-\Gamma L_+ - \Delta T_+} = \sum_{n=1}^{\infty} \mathrm{E}e^{-\Gamma(p\tau_n - S_n) - \Delta\tau_n} 1_{(N_+ = n)}.$$

Clearly, for every $n = 1, 2, \ldots$,

$$1_{(N_+ = n)} = 1_{(p\tau_i - S_i < 0, i = 1, \ldots, n-1)} - 1_{(p\tau_i - S_i < 0, i = 1, \ldots, n)},$$

where $1_{(p\tau_i - S_i < 0, i = 1, \ldots, n-1)} = 1$ for $n = 1$. Hence,

$$\begin{aligned} &\mathrm{E}e^{-\Gamma L_+ - \Delta T_+} \\ &= \mathrm{E}e^{\Gamma X - (\Delta + p\Gamma)\tau} - \left(1 - \mathrm{E}e^{\Gamma X - (\Delta + p\Gamma)\tau}\right) \sum_{n=1}^{\infty} \mathrm{E}e^{-\Gamma(p\tau_n - S_n) - \Delta\tau_n} 1_{(p\tau_i - S_i < 0, i = 1, \ldots, n)}. \end{aligned} \tag{4.9}$$

By the duality principle (see, e.g. page 378 of Feller 1971), for every $n = 1, 2, \ldots$,

$$\mathrm{E}e^{-\Gamma(p\tau_n - S_n) - \Delta\tau_n} 1_{(p\tau_i - S_i < 0, i = 1, \ldots, n)} = \mathrm{E}e^{-\Gamma(p\tau_n - S_n) - \Delta\tau_n} 1_{(p\tau_n - S_n < p\tau_i - S_i, i = 1, \ldots, n-1, p\tau_n - S_n < 0)}.$$

The joint event in the indicator function on the right-hand side above implies that τ_n is a moment of descending ladder. Denote by ν_j the number of innovations needed for exactly j descending ladder epochs, $j = 1, 2, \ldots$. Hence,

$$1_{(p\tau_n - S_n < p\tau_i - S_i, i = 1, \ldots, n-1, p\tau_n - S_n < 0)} = \sum_{j=1}^{n} 1_{(\nu_j = n)}.$$

Plug these identities into the sum on the right-hand side of (4.9), then interchange the order of summation and eliminate the certain event $(\nu_j \geq j)$. We have

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{E}\mathrm{e}^{-\Gamma(p\tau_n - S_n) - \Delta\tau_n} 1_{(p\tau_i - S_i < 0, i=1,\ldots,n)} &= \sum_{n=1}^{\infty}\sum_{j=1}^{n} \mathrm{E}\mathrm{e}^{-\Gamma\left(p\tau_{\nu_j} - S_{\nu_j}\right) - \Delta\tau_{\nu_j}} 1_{(\nu_j = n)} \\ &= \sum_{j=1}^{\infty} \mathrm{E}\mathrm{e}^{-\Gamma\left(p\tau_{\nu_j} - S_{\nu_j}\right) - \Delta\tau_{\nu_j}} \\ &= \sum_{j=1}^{\infty} \mathrm{E}\mathrm{e}^{\Gamma\sum_{i=1}^{j} L_{i-} - \Delta\sum_{i=1}^{j} T_{i-}} \\ &= \frac{\mathrm{E}\mathrm{e}^{\Gamma L_- - \Delta T_-}}{1 - \mathrm{E}\mathrm{e}^{\Gamma L_- - \Delta T_-}},\end{aligned}$$

where (L_{i-}, T_{i-}), $i = 1, 2, \ldots$, are i.i.d. copies of (L_-, T_-). Plugging this into (4.9) yields

$$\mathrm{E}\mathrm{e}^{-\Gamma L_+ - \Delta T_+} = \mathrm{E}\mathrm{e}^{\Gamma X - (\Delta + p\Gamma)\tau} - \left(1 - \mathrm{E}\mathrm{e}^{\Gamma X - (\Delta + p\Gamma)\tau}\right) \frac{\mathrm{E}\mathrm{e}^{\Gamma L_- - \Delta T_-}}{1 - \mathrm{E}\mathrm{e}^{\Gamma L_- - \Delta T_-}},$$

which, upon some obvious rearrangement, gives relation (4.8). □

The next two lemmas give some easy identities which are useful in proving Theorem 3.1.

Lemma 4.7. *Let $\delta \geq 0$, γ and b be real numbers. All expectations appearing below are assumed to be finite. Then for $\gamma \neq b$, it holds that*

$$\iiint \mathrm{e}^{by + \gamma z} \frac{f(x + y + z)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-} - \mathrm{E}\mathrm{e}^{bL_- - \delta T_-}}{\gamma - b},$$

while for $\gamma = b$, it holds that

$$\iiint \mathrm{e}^{\gamma(y+z)} \frac{f(x + y + z)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathrm{E}L_-\mathrm{e}^{\gamma L_- - \delta T_-}.$$

Proof. By the definition of $J_\delta(x|0)$ and relation (2.3), we have

$$\begin{aligned}\iiint \mathrm{e}^{by + \gamma z} \frac{f(x + y + z)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiiint \mathrm{e}^{by + \gamma z - \delta t} J(x, y + z, t|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &= \iiint_z^\infty \mathrm{e}^{b(y-z) + \gamma z - \delta t} J(y, t|0)\mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &= \iint\left(\int_0^y \mathrm{e}^{(\gamma - b)z}\mathrm{d}z\right)\mathrm{e}^{by - \delta t} J(y, t|0)\mathrm{d}y\mathrm{d}t.\end{aligned}$$

From this expression the two formulas follow easily. □

Lemma 4.8. *For $\delta \geq 0$, $-\infty < \gamma < \infty$, and $\delta + p\gamma > 0$,*

$$\int \mathrm{e}^{-\gamma x} \frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x = \frac{\mathrm{E}\mathrm{e}^{-(\delta + p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+}}.$$

Proof. By definition we have

$$\int \mathrm{e}^{-\gamma x} \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x = \iint \frac{\mathrm{e}^{-\gamma x-\delta t}}{\overline{F}(x)} J(x,t|0)\mathrm{d}x\mathrm{d}t. \tag{4.10}$$

According to the number of innovations needed for the first descending ladder, we expand $J(x,t|0)\mathrm{d}x\mathrm{d}t$ as

$$\begin{aligned} &\sum_{n=0}^{\infty} \Pr\left(p\tau_i - S_i \ge 0, i=1,\ldots,n, p\tau_{n+1} - S_{n+1} < 0; p\tau_{n+1} - S_n \in \mathrm{d}x, \tau_{n+1} \in \mathrm{d}t\right) \\ &= \overline{F}(x) \sum_{n=0}^{\infty} \Pr\left(p\tau_i - S_i \ge 0, i=1,\ldots,n; p\tau_{n+1} - S_n \in \mathrm{d}x, \tau_{n+1} \in \mathrm{d}t\right). \end{aligned}$$

The first probability above corresponding to $n=0$ reduces to $\Pr\left(p\tau \in \mathrm{d}x, \tau \in \mathrm{d}t\right)$, which is concentrated on the line $pt = x$. We follow the same approach as in the proof of Lemma 4.6 to deal with the other probabilities. For each $n = 1, 2, \ldots$, by the duality principle,

$$\begin{aligned} &\Pr\left(p\tau_i - S_i \ge 0, i=1,\ldots,n; p\tau_{n+1} - S_n \in \mathrm{d}x, \tau_{n+1} \in \mathrm{d}t\right) \\ &= \Pr\left(p\tau_n - S_n \ge p\tau_i - S_i, i=1,\ldots,n-1, p\tau_n - S_n \ge 0; p\tau_{n+1} - S_n \in \mathrm{d}x, \tau_{n+1} \in \mathrm{d}t\right). \end{aligned}$$

The joint event in the probability on the right-hand side of the above indicates that τ_n is a moment of ascending ladder. Denote by ν_j the number of innovations needed for exactly j ascending ladder epochs, $j = 1, 2, \ldots$. Thus, this probability can be rewritten as

$$\sum_{j=1}^{n} \Pr\left(\nu_j = n; p\tau_{\nu_j} - S_{\nu_j} + p\tau^* \in \mathrm{d}x, \tau_{\nu_j} + \tau^* \in \mathrm{d}t\right),$$

where τ^* is a random variable equal in distribution to τ and independent of all the other sources of randomness. Plug, in turn, all these identities into (4.10), then interchange the order of summation and eliminate the certain event $(\nu_j \ge j)$. We obtain

$$\begin{aligned} &\int \mathrm{e}^{-\gamma x} \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x \\ &= \mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau} + \sum_{j=1}^{\infty} \iint \mathrm{e}^{-\gamma x-\delta t} \Pr\left(p\tau_{\nu_j} - S_{\nu_j} + p\tau^* \in \mathrm{d}x, \tau_{\nu_j} + \tau^* \in \mathrm{d}t\right) \\ &= \mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau} + \sum_{j=1}^{\infty} \iint \mathrm{e}^{-\gamma x-\delta t} \Pr\left(\sum_{i=1}^{j} L_{i+} + p\tau^* \in \mathrm{d}x, \sum_{i=1}^{j} T_{i+} + \tau^* \in \mathrm{d}t\right), \end{aligned}$$

where (L_{i+}, T_{i+}), $i = 1, 2, \ldots$, are i.i.d. copies of (L_+, T_+) and are independent of τ^*. Hence,

$$\int \mathrm{e}^{-\gamma x} \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x = \mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau} \sum_{j=0}^{\infty} \left(\mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+}\right)^j = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+}}.$$

Actually, in the last step above we tacitly used the fact $\mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+} < 1$, which trivially holds when $\gamma > 0$ and can be easily verified when $\gamma \le 0$ by using $L_+ \le pT_+$. This ends the proof of Lemma 4.8. □

5 Proof of Theorem 3.1

5.1 Asymptotics for the function v

Introduce a renewal measure on $[0, \infty)$ associated with the ascending ladder heights as

$$R_+ = \sum_{n=0}^{\infty} H_+^{n*}.$$

Lemma 5.1. *Recall the functions k and v introduced in Subsection 2.2. Assume that f is bounded and* $\mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-} < 1$.

(1) When $\delta \geq 0$, $\gamma \geq 0$, and $\delta \vee \gamma > 0$, further assume $f \in \mathcal{S}_d(\gamma)$. Then

$$\lim_{u\to\infty} \frac{v(u)}{f(u)} = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{\gamma X - (\delta+p\gamma)\tau}} \frac{1}{1 - \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-}}. \tag{5.1}$$

(2) When $\delta = \gamma = 0$, further assume that $\overline{F} \in \mathcal{S}_d(0)$ and f is eventually non-increasing. Then

$$\lim_{u\to\infty} \frac{v(u)}{\overline{F}(u)} = \frac{1}{\rho\,(1 - \psi(0))}. \tag{5.2}$$

(3) The functions k and v are bounded, hence are locally integrable.

Proof. Since v is expressed as an infinite sum of convolutions of k, we first deal with k. When $\delta > 0$ and $\gamma \geq 0$, we have

$$\lim_{u\to\infty} \frac{k(u)}{f(u)} = \lim_{u\to\infty} \int \frac{f(x+u)}{f(u)} \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x = \int \mathrm{e}^{-\gamma x} \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x = \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+}}, \tag{5.3}$$

where the second step is due to the dominated convergence theorem and the last step is due to Lemma 4.8. However, to be completely right we must verify the usage of the dominated convergence theorem. By Lemma 4.4(1), $\overline{F} \in \mathcal{L}_d(\gamma)$. For arbitrarily chosen $\varepsilon > 0$ satisfying $2p\varepsilon < \delta$, by Lemma 4.1(1, 2), it is easy to see that there is some constant $c > 0$ such that, for all large $u > 0$ and all $x > 0$,

$$\frac{f(x+u)}{f(u)} \frac{J_\delta(x|0)}{\overline{F}(x)} \leq c\mathrm{e}^{2\varepsilon x} J_\delta(x|0).$$

This upper bound is integrable, as,

$$\begin{aligned} \int \mathrm{e}^{2\varepsilon x} J_\delta(x|0)\mathrm{d}x &= \iint \mathrm{e}^{2\varepsilon x - \delta t} J(x, t|0)\mathrm{d}x\mathrm{d}t \\ &\leq \iint \mathrm{e}^{2\varepsilon x - \delta t} \sum_{n=1}^{\infty} \Pr\left(pt - S_{n-1} \in \mathrm{d}x\right) \Pr\left(\tau_n \in \mathrm{d}t\right) \\ &= \sum_{n=1}^{\infty} \left(\mathrm{E}\mathrm{e}^{-2\varepsilon X}\right)^{n-1} \left(\mathrm{E}\mathrm{e}^{-(\delta-2p\varepsilon)\tau}\right)^n < \infty. \end{aligned}$$

When $\delta = 0$, by (2.3) it is clear that

$$k(u) = \iint \frac{f(x+u)}{\overline{F}(x)} J(x,t|0)\mathrm{d}x\mathrm{d}t = \iint J(x,u,t|0)\mathrm{d}x\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}u}\Pr\left(L_- > u\right).$$

We use the identity

$$\Pr\left(L_- > u\right) = \int_{0-}^{\infty} \Pr\left(X - p\tau > x + u\right) R_+(\mathrm{d}x), \qquad u \geq 0,$$

which is a direct consequence of (4.6); see also relation (6.4.10) of Rolski et al. (1999) or page 262 of Asmussen (2000). This gives that

$$k(u) = \iint_{0-}^{\infty} f(x+u+pt) R_+(\mathrm{d}x) G(\mathrm{d}t). \tag{5.4}$$

Hence for $\delta = 0$ and $\gamma > 0$, by the dominated convergence theorem justified by Lemma 4.1(1) we have

$$\lim_{u\to\infty} \frac{k(u)}{f(u)} = \iint_{0-}^{\infty} \mathrm{e}^{-\gamma(x+pt)} R_+(\mathrm{d}x) G(\mathrm{d}t) = \frac{\mathrm{E}\mathrm{e}^{-p\gamma\tau}}{1 - \mathrm{E}\mathrm{e}^{-\gamma L_+}}, \tag{5.5}$$

showing that relation (5.3) still holds. For $\delta = \gamma = 0$, applying Lemma 4.5 to (5.4) yields

$$k(u) \sim \frac{1}{\mathrm{E}L_+} \iint f(u+x+pt)\mathrm{d}x G(\mathrm{d}t) = \frac{1}{\mathrm{E}L_+} \int \overline{F}(u+pt) G(\mathrm{d}t) \sim \frac{\overline{F}(u)}{\mathrm{E}L_+}, \tag{5.6}$$

where in the last step we used the dominated convergence theorem justified by $\overline{F} \in \mathcal{L}_d(0)$.

So far, we have proved that when $\delta \geq 0$, $\gamma \geq 0$, and $\delta \vee \gamma > 0$ relation (5.3) holds, while when $\delta = \gamma = 0$ relation (5.6) holds. Before proving the three items of Lemma 5.1, we notice that, for $\delta \geq 0$ and $\gamma \geq 0$, by relation (2.3),

$$\hat{k}(\gamma) = \iint \mathrm{e}^{\gamma y} \frac{f(x+y)}{\overline{F}(x)} J_\delta(x|0)\mathrm{d}x\mathrm{d}y = \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-} < 1. \tag{5.7}$$

We distribute the verification of the boundedness of k and v to the proofs of (1) and (2).

(1) For this case, it is easy to verify the boundedness of k. Actually, by Lemma 4.8, it holds for all $u > 0$ that

$$k(u) \leq \sup_{0<x<\infty} f(x) \int \frac{J_\delta(x|0)}{\overline{F}(x)} \mathrm{d}x = \sup_{0<x<\infty} f(x) \frac{\mathrm{E}\mathrm{e}^{-\delta\tau}}{1 - \mathrm{E}\mathrm{e}^{-\delta T_+}} < \infty.$$

By Lemma 4.2(2) and relation (5.7), the boundedness of v is an immediate consequence of the boundedness of k.

Now we derive the asymptotic formula (5.1) for v. Applying the dominated convergence theorem justified by Lemma 4.2, then using relations (5.3) and (5.7),

$$\begin{aligned} v(u) &\sim k(u) \sum_{n=1}^{\infty} n \left(\hat{k}(\gamma)\right)^{n-1} \\ &\sim \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{-\gamma L_+ - \delta T_+}} \frac{1}{\left(1 - \hat{k}(\gamma)\right)^2} f(u) \\ &= \frac{\mathrm{E}\mathrm{e}^{-(\delta+p\gamma)\tau}}{1 - \mathrm{E}\mathrm{e}^{\gamma X - (\delta+p\gamma)\tau}} \frac{1}{1 - \mathrm{E}\mathrm{e}^{\gamma L_- - \delta T_-}} f(u), \end{aligned}$$

where in the last step we used Lemma 4.6.

(2) Define $\tilde{f}(x) = \sup_{y>x} f(y)$ for $x \geq 0$. Clearly, $\tilde{f}$ is bounded by $\tilde{f}(0)$, non-increasing on $(0, \infty)$, and identical to f for all large x. Starting from (5.4) and following the proof of (4.5), we can obtain that, for arbitrarily fixed small $\varepsilon > 0$, some large $c > 0$, and all $u > 0$,

$$\begin{aligned} k(u) &\leq \int \left(\int_{0-}^{\infty} \tilde{f}(x+u+pt) R_+(\mathrm{d}x) \right) G(\mathrm{d}t) \\ &\leq \int \left(\tilde{f}(u+pt) R_+[0,c] + \frac{1+\varepsilon}{\mathrm{E}L_+} \int_{u+pt+c-1}^{\infty} \tilde{f}(y)\mathrm{d}y \right) G(\mathrm{d}t) \\ &< \infty. \end{aligned} \tag{5.8}$$

Relation (5.6) and the assumption $\overline{F} \in \mathcal{S}_d(0)$ imply $k \in \mathcal{S}_d(0)$. Thus, as explained in the proof of (1), v is bounded.

Now we derive the asymptotic formula (5.2) for v. Similarly as in the proof of (1), applying the dominated convergence theorem justified by Lemma 4.2, then using relations (5.6) and (5.7) with $\gamma = 0$,

$$v(u) \sim k(u) \sum_{n=1}^{\infty} n \left(\hat{k}(0)\right)^{n-1} \sim \frac{\overline{F}(u)}{\mathrm{E}L_+ (1-\psi(0))^2} = \frac{\overline{F}(u)}{\rho (1-\psi(0))},$$

where the last step is due to (4.7). □

5.2 Asymptotics for the function h

Lemma 5.2. *Recall the functions w and h introduced in Subsection 2.2.*

(1) When $\delta \geq 0$, $\alpha \geq 0$, and $\delta \vee \alpha > 0$, assume $w \in \mathcal{L}_d(\alpha)$. Then

$$\lim_{u\to\infty} \frac{h(u)}{w(u)} = \frac{\mathrm{E}e^{-(\delta+p\alpha)\tau}}{1-\mathrm{E}e^{-\alpha L_+ - \delta T_+}}. \tag{5.9}$$

Furthermore, if w is locally integrable, so is h.

(2) When $\delta = \alpha = 0$, assume that w is eventually non-increasing such that $\overline{W}(u) < \infty$ for all large $u > 0$ and $\overline{W} \in \mathcal{L}_d(0)$. Then

$$\lim_{u\to\infty} \frac{h(u)}{\overline{W}(u)} = \frac{1}{\mathrm{E}L_+}. \tag{5.10}$$

Furthermore, if w is globally integrable then h is locally integrable.

Proof. Recall relation (2.7). When $\delta > 0$ and $\alpha \geq 0$, the proof of relation (5.9) is the same as the proof of relation (5.3). When $\delta = 0$, it follows from Pitts and Politis (2007) that

$$J_0(x|0) = \overline{F}(x) \int_{0-}^{x} \frac{1}{p} g\left(\frac{x-y}{p}\right) R_+(\mathrm{d}y). \tag{5.11}$$

Plugging (5.11) into (2.7), then interchanging the order of integration twice, we obtain

$$
\begin{aligned}
h(u) &= \int w(x+u)\int_{0-}^{x}\frac{1}{p}g\left(\frac{x-y}{p}\right)R_+(\mathrm{d}y)\mathrm{d}x \\
&= \int_{0-}^{\infty}\int w(x+y+u)\frac{1}{p}g\left(\frac{x}{p}\right)\mathrm{d}xR_+(\mathrm{d}y) \\
&= \iint_{0-}^{\infty} w(px+y+u)R_+(\mathrm{d}y)G(\mathrm{d}x). \qquad (5.12)
\end{aligned}
$$

When $\delta = 0$ and $\alpha > 0$, the same proof as in (5.5) can be used to show that relation (5.9) still holds. When $\delta = \alpha = 0$, by Lemma 4.5,

$$
h(u) \sim \frac{1}{\mathrm{E}L_+}\iint w(px+y+u)\mathrm{d}yG(\mathrm{d}x) \sim \frac{1}{\mathrm{E}L_+}\overline{W}(u).
$$

This proves relation (5.10).

It remains to verify the integrability of h. Consider case (1) where w is locally integrable. Let $u_0 > 0$ be arbitrarily fixed. Clearly, there is some small $\varepsilon > 0$ such that $\delta + p(\alpha - \varepsilon) > 0$. By Lemma 4.1(2), for this ε there are some positive constants c and x_0 such that, for all $x \ge x_0$,

$$
\int_0^{u_0} w(x+u)\mathrm{d}u \le c\mathrm{e}^{-(\alpha-\varepsilon)x}. \qquad (5.13)
$$

We start from (2.7) to deal with $\int_0^{u_0} h(u)\mathrm{d}u$ and we split it into two parts as

$$
\int_0^{u_0} h(u)\mathrm{d}u = \left(\int_0^{x_0} + \int_{x_0}^{\infty}\right)\left(\int_0^{u_0} w(x+u)\mathrm{d}u\right)\frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x.
$$

The first term above is clearly finite since w is locally integrable. Using (5.13), the second term is bounded by

$$
c\int_{x_0}^{\infty} \mathrm{e}^{-(\alpha-\varepsilon)x}\frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x,
$$

which is finite by Lemma 4.8 since $\delta + p(\alpha - \varepsilon) > 0$. This proves the local integrability of h.

Consider case (2) where w is globally integrable. For every $u_0 > 0$, we use (5.12) to obtain

$$
\int_0^{u_0} h(u)\mathrm{d}u = \iint_{0-}^{\infty}\left(\int_0^{u_0} w(px+y+u)\mathrm{d}u\right)R_+(\mathrm{d}y)G(\mathrm{d}x).
$$

Note that $\int_0^{u_0} w(z+u)\mathrm{d}u$, as a function of z, is eventually non-increasing and globally integrable since w is so. Then, as in (5.8), we follow the proof of (4.5) to obtain the finiteness of $\int_0^{u_0} h(u)\mathrm{d}u$. Hence, h is locally integrable. □

5.3 Proof of Theorem 3.1

We start from (2.5) and use Lemma 4.3 to derive asymptotics for ϕ. Note that under the conditions of Theorem 3.1, by Lemmas 5.1 and 5.2, both h and v are locally integrable. Trivially,

$$
\hat{h}(\gamma) = \iint \mathrm{e}^{\gamma z}w(x+z)\frac{J_\delta(x|0)}{\overline{F}(x)}\mathrm{d}x\mathrm{d}z. \qquad (5.14)
$$

In particular, when $\delta=\gamma=0$,

$$\hat{h}(0)=\iiint \varpi(x+z,y)J(x,y+z|0)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{5.15}$$

It is easy to verify the finiteness of $\hat{h}(\gamma)$ whenever necessary. By relation (5.7), we have

$$\hat{v}(\gamma)=\sum_{n=1}^{\infty}\left(\hat{k}(\gamma)\right)^n=\frac{\mathrm{E}e^{\gamma L_- -\delta T_-}}{1-\mathrm{E}e^{\gamma L_- -\delta T_-}}. \tag{5.16}$$

(1) Comparing (5.9) with (5.1), then using Lemmas 4.1(2) and 4.3(1), we have

$$\phi(u)\sim h(u)\left(\hat{v}(\alpha)+1\right).$$

Plugging (5.9) and (5.16) (with γ replaced by α) into the above, then using Lemma 4.6, we obtain (3.1).

(2) Similarly as above, by Lemmas 4.1(2) and 4.3(1) we have

$$\phi(u)\sim v(u)\hat{h}(\gamma).$$

Plugging (5.1) and (5.14) into the above, we obtain (3.2).

(3) Similarly as above, by Lemma 4.3(2), it is always true that

$$\phi(u)\sim h(u)\left(\hat{v}(\gamma)+1\right)+v(u)\hat{h}(\gamma).$$

Plugging (5.9), (5.16), (5.1), and (5.14) into the above and using Lemma 4.6, we obtain (3.3).

(4) Comparing (5.10) with (5.1) and using Lemmas 4.1(2) and 4.3(1), it holds that

$$\phi(u)\sim h(u)\left(\hat{v}(0)+1\right). \tag{5.17}$$

Plugging $\delta=\gamma=0$ to (5.16) yields

$$\hat{v}(0)=\frac{\psi(0)}{1-\psi(0)}. \tag{5.18}$$

Then, plugging (5.10) and (5.18) into (5.17) and using (4.7), we obtain (3.5).

(5) Comparing (5.9) with (5.2) and using Lemmas 4.1(2) and 4.3(1), it holds that

$$\phi(u)\sim v(u)\hat{h}(0).$$

Plugging (5.2) and (5.15) into the above, we obtain (3.6).

(6) Comparing (5.10) with (5.2) then using Lemma 4.3(2), it holds that

$$\phi(u)\sim h(u)\left(\hat{v}(0)+1\right)+v(u)\hat{h}(0).$$

Plugging (5.10), (5.18), (5.2), and (5.15) into the above then using (4.7), we obtain (3.7).□

6 Numerical results

To examine the accuracy of the asymptotic formulas obtained in this paper, we consider the compound Poisson risk model in which the inter-arrival time distribution G is exponential with mean $1/\lambda$ and the claim-size distribution F is inverse Gaussian with density function f given in (2.14). In this case, $\gamma = \beta/(2\mu^2)$ and $\hat{f}(\gamma) = e^{\beta\mu^{-1}}$. Hence, the condition $\hat{f}(\gamma) < 1 + (p\gamma + \delta)/\lambda$ reduces to

$$\lambda e^{\beta\mu^{-1}} < \lambda + 0.5p\beta\mu^{-2} + \delta.$$

Choose $\mu = 1$ and $\beta = 0.2$, so that

$$f(x) = \left(10\pi x^3\right)^{-0.5} e^{-0.1x-0.1x^{-1}+0.2}.$$

Thus, $f \in \mathcal{S}_d(\gamma)$ with $\gamma = 0.1$. Further choose $\lambda = 0.5$, $p = 0.6$, $\delta = 0.06$. Solving equation (3.18) leads to $\theta = 0.126864$. We test the following four cases:

(1) Choose $\varpi(x, y) = e^{0.1x}$. It follows from (2.6) that $w(u) = e^{0.1u}\overline{F}(u)$, so that $w \in \mathcal{L}_d(0)$. This applies to the situation of Corollary 3.2(1). Hence,

$$\phi(u) \sim 1.81595e^{0.1u} \int_u^\infty x^{-1.5} e^{-0.1x-0.1x^{-1}} dx.$$

(2) Choose $\varpi(x, y) = \delta^{-1}\left(1 - e^{-\theta y}\right) = 0.06^{-1}\left(1 - e^{-0.126864y}\right)$. For this case, Gerber and Shiu (1998) interpreted $\phi(u)$ as the expected present value of a deferred continuous annuity at a rate of 1 per unit time, starting at the time of ruin and ending as soon as the surplus rises to zero. It follows from (2.6) that

$$w(u) \sim \frac{50}{3} f(u) \int \left(e^{-0.1y} - e^{-0.226864y}\right) dy = 20.3098u^{-1.5}e^{-0.1u-0.1u^{-1}},$$

so that $w \in \mathcal{L}_d(\alpha)$ with $\alpha = 0.1$. This applies to the situation of Corollary 3.2(3). Hence,

$$\phi(u) \sim 7967.49u^{-1.5}e^{-0.1u-0.1u^{-1}}.$$

(3) Choose $\varpi(x, y) = (1 - e^{1-y}) \vee 0$, which Gerber and Shiu (1998) interpreted as the payoff at exercise of a perpetual American put option with an exercise price 1 and an option-exercise boundary e. It follows from (2.6) that

$$w(u) \sim f(u) \int_1^\infty \left(e^{-0.1y} - e^{1-1.1y}\right) dy = 1.79251u^{-1.5}e^{-0.1u-0.1u^{-1}},$$

so that $w \in \mathcal{L}_d(\alpha)$ with $\alpha = 0.1$. This also applies to the situation of Corollary 3.2(3). Hence,

$$\phi(u) \sim 751.675u^{-1.5}e^{-0.1u-0.1u^{-1}}.$$

(4) Choose $\varpi(x,y) \equiv 1$. For this case, $\phi(u)$ reduces to the Laplace transform of the time of ruin. It follows from (2.6) that $w = \overline{F}$, so that $w \in \mathcal{L}_d(\alpha)$ with $\alpha = 0.1$. This again applies to the situation of Corollary 3.2(3). Hence,

$$\phi(u) \sim 103.397 \int_u^\infty x^{-1.5} \mathrm{e}^{-0.1x - 0.1x^{-1}} \mathrm{d}x.$$

We take advantage of the packages Mathematica 6.0 and Matlab 6.5 to examine the accuracy of these asymptotic formulas. The numerical results are copied to Tables 1–4. Furthermore, the ratios and relative errors in these tables are plotted in Figures 1–4 accordingly. All of these tables and figures show a quite satisfactory accuracy of the obtained asymptotic formulas with u relatively large. However, we need to point out that the accuracy might not be as good if we consider the case of heavy-tailed claim-size distributions for which $\gamma = 0$.

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Table 1: Numerical Results for Case 1

u	a=asymptotics	s=simulation	a/s	|s-a|/s
10	1.14348	1.13678	1.00589384	0.00589384
20	0.628866	0.625181	1.005894293	0.005894293
30	0.414718	0.412288	1.005893938	0.005893938
40	0.300661	0.2989	1.005891603	0.005891603
50	0.231176	0.229821	1.005895893	0.005895893
60	0.185055	0.18397	1.005897701	0.005897701
70	0.152555	0.151661	1.005894726	0.005894726
80	0.128617	0.127864	1.00588907	0.00588907
90	0.110374	0.109727	1.005896452	0.005896452
100	0.0960869	0.0955238	1.005894866	0.005894866
110	0.0846476	0.0841516	1.005894124	0.005894124
120	0.0753179	0.0748766	1.005893697	0.005893697
130	0.0675894	0.0671933	1.005894933	0.005894933
140	0.0611012	0.0607431	1.00589532	0.00589532
150	0.0555908	0.0552651	1.005893412	0.005893412
160	0.0508634	0.0505654	1.005893358	0.005893358
170	0.0467713	0.0464973	1.005892815	0.005892815
180	0.0432009	0.0429478	1.005893201	0.005893201
190	0.0400635	0.0398287	1.005895246	0.005895246
200	0.0372888	0.0370703	1.005894206	0.005894206

Table 2: Numerical Results for Case 2

u	a=asymptotics	s=simulation	a/s	|s-a|/s
10	30.3982	18.5162	1.641708342	0.641708342
20	5.2112	3.74622	1.391055517	0.391055517
30	1.15864	0.908859	1.274829209	0.274829209
40	0.292558	0.242397	1.20693738	0.20693738
50	0.0796835	0.0685644	1.162170164	0.162170164
60	0.0228228	0.0201912	1.130334007	0.130334007
70	0.00677551	0.00612344	1.10648753	0.10648753
80	0.00206623	0.00189922	1.0879361	0.0879361
90	0.0006434	0.00059958	1.073084492	0.073084492
100	0.000203719	0.000192022	1.060914895	0.060914895
110	6.53899E-05	6.22311E-05	1.050759186	0.050759186
120	2.12289E-05	2.03703E-05	1.0421496	0.0421496
130	6.95863E-06	6.72485E-06	1.034763601	0.034763601
140	2.29985E-06	2.23645E-06	1.028348499	0.028348499
150	7.6556E-07	7.48546E-07	1.022729398	0.022729398
160	2.56433E-07	2.51957E-07	1.017764936	0.017764936
170	8.637E-08	8.52327E-08	1.01334347	0.01334347
180	2.92335E-08	2.89618E-08	1.009381323	0.009381323
190	9.93813E-09	9.88068E-09	1.005814377	0.005814377
200	3.39189E-09	3.38316E-09	1.002580428	0.002580428

Table 3: Numerical Results for Case 3

u	a=asymptotics	s=simulation	a/s	|s-a|/s
10	2.86785	1.73275	1.655085846	0.655085846
20	0.491639	0.350542	1.402510969	0.402510969
30	0.109309	0.0850405	1.285375792	0.285375792
40	0.0276008	0.0226802	1.216955759	0.216955759
50	0.00751756	0.00641521	1.171833814	0.171833814
60	0.00215317	0.00188917	1.139743909	0.139743909
70	0.00063922	0.000572927	1.115709331	0.115709331
80	0.000194934	0.000177695	1.097014547	0.097014547
90	6.07002E-05	5.60979E-05	1.082040504	0.082040504
100	1.92194E-05	1.79659E-05	1.069771066	0.069771066
110	6.16906E-06	5.82241E-06	1.059537202	0.059537202
120	2.0028E-06	1.90586E-06	1.050864177	0.050864177
130	6.56496E-07	6.29181E-07	1.043413581	0.043413581
140	2.16974E-07	2.09243E-07	1.036947473	0.036947473
150	7.22251E-08	7.00342E-08	1.031283287	0.031283287
160	2.41926E-08	2.35731E-08	1.026279955	0.026279955
170	8.14838E-09	7.97437E-09	1.02182116	0.02182116
180	2.75797E-09	2.70966E-09	1.017828805	0.017828805
190	9.3759E-10	9.24433E-10	1.014232508	0.014232508
200	3.2E-10	3.16527E-10	1.010972208	0.010972208

Table 4: Numerical Results for Case 4

u	a=asymptotics	s=simulation	a/s	|s-a|/s
10	2.25144	3.67434	0.612746779	0.387253221
20	0.455506	0.630011	0.723012773	0.276987227
30	0.110508	0.140089	0.788841379	0.211158621
40	0.029473	0.0353754	0.83314959	0.16685041
50	0.00833671	0.00963563	0.865196152	0.134803848
60	0.00245504	0.00275993	0.889529807	0.110470193
70	0.000744544	0.000819379	0.908668638	0.091331362
80	0.000230924	0.000249881	0.924135889	0.075864111
90	7.29023E-05	7.78117E-05	0.936906661	0.063093339
100	2.33477E-05	2.46379E-05	0.947633524	0.052366476
110	7.5666E-06	7.90839E-06	0.956781342	0.043218658
120	2.47679E-06	2.5675E-06	0.964669912	0.035330088
130	8.17665E-07	8.41612E-07	0.971546271	0.028453729
140	2.71927E-07	2.78159E-07	0.977595548	0.022404452
150	9.10147E-08	9.25924E-08	0.982960805	0.017039195
160	3.06351E-08	3.10151E-08	0.987747903	0.012252097
170	1.03633E-08	1.04464E-08	0.992045106	0.007954894
180	3.52142E-09	3.53579E-09	0.995935845	0.004064155
190	1.20138E-09	1.20202E-09	0.999467563	0.000532437
200	4.11353E-10	4.10254E-10	1.002678828	0.002678828

Figure 1: Ratio and Relative Error Curves of Asymptotic and Simulated Values in Case 1

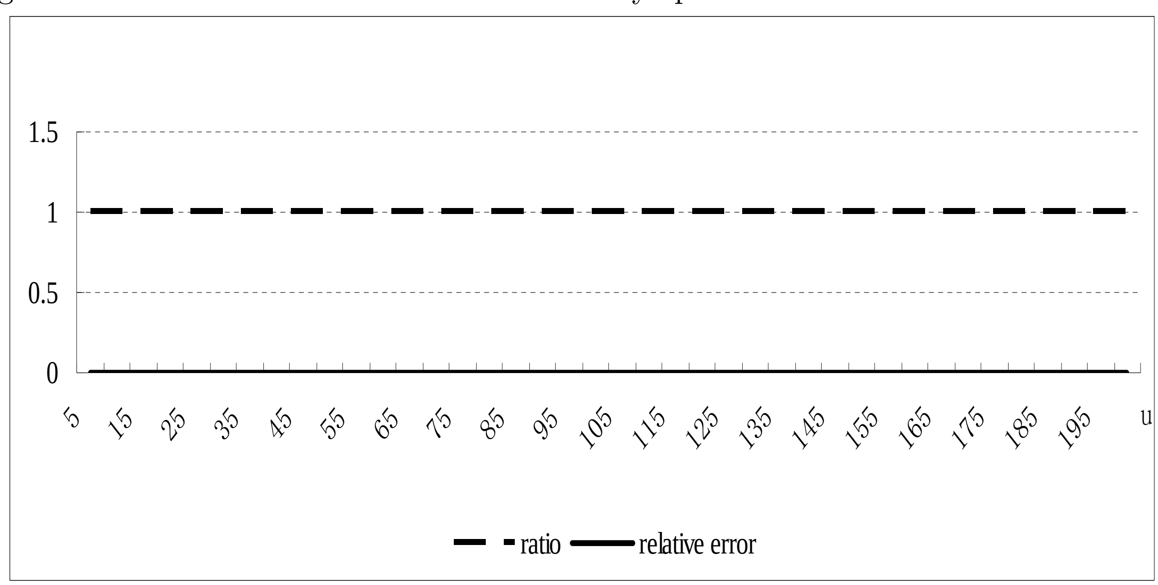


Figure 2: Ratio and Relative Error Curves of Asymptotic and Simulated Values in Case 2

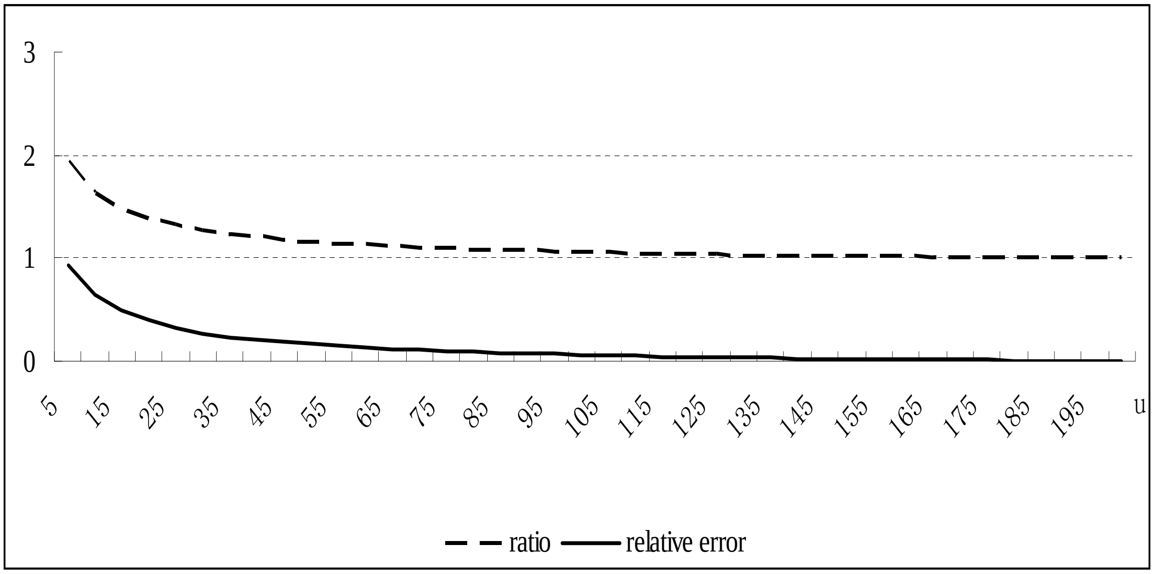


Figure 3: Ratio and Relative Error Curves of Asymptotic and Simulated Values in Case 3

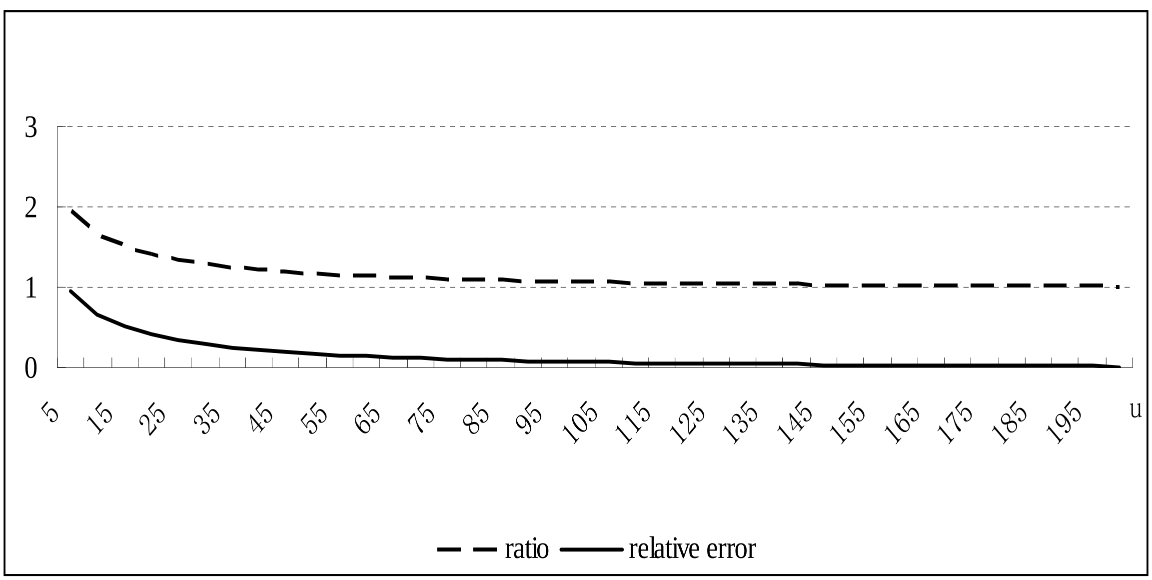
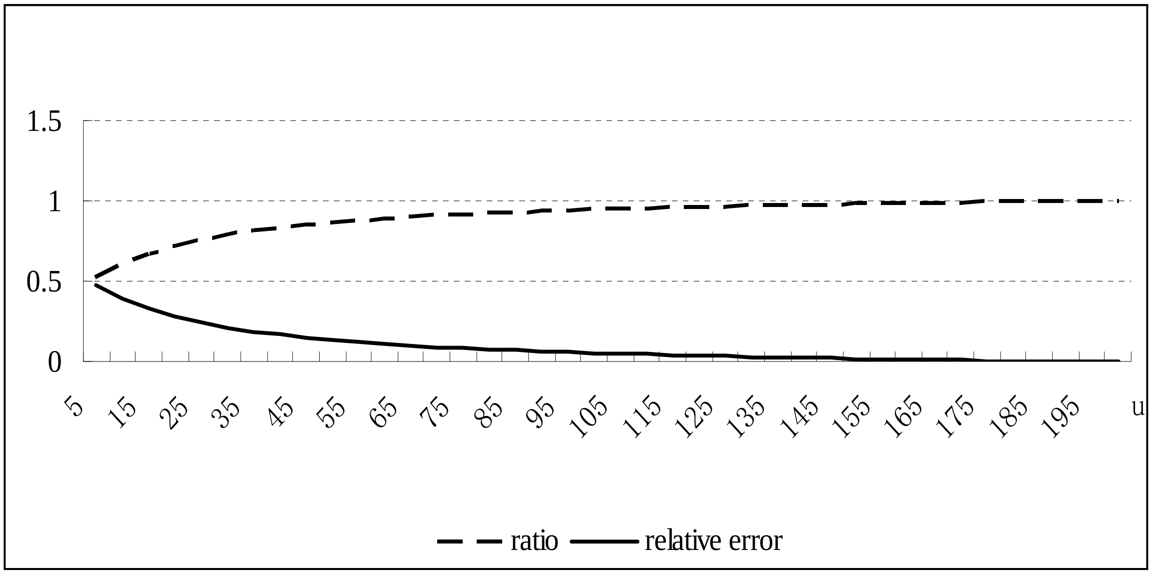


Figure 4: Ratio and Relative Error Curves of Asymptotic and Simulated Values in Case 4



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