Asymptotic Tail Probabilities of Sums of Dependent Subexponential Random Variables

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Abstract

In this paper we study the asymptotic behavior of the tail probabilities of sums of dependent and real-valued random variables whose distributions are assumed to be subexponential and not necessarily of dominated variation. We propose two general dependence assumptions under which the asymptotic behavior of the tail probabilities of the sums is the same as that in the independent case. In particular, the two dependence assumptions are satisfied by multivariate Farlie-Gumbel-Morgenstern distributions.

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1 Introduction

Throughout the paper, denote by $F_1 * \cdots * F_n$ the convolution of distributions $F_1, \ldots, F_n$ and by F^{*n} the n-fold convolution of a distribution F. Limits and asymptotic relations are assumed to be for $x \to \infty$ unless mentioned otherwise. For two positive functions $u(\cdot)$ and $v(\cdot)$, we write $u(x) \sim v(x)$ if $\lim u(x)/v(x) = 1$, write $u(x) \lesssim v(x)$ if $\limsup u(x)/v(x) \leq 1$, and write $u(x) \gtrsim v(x)$ if $\liminf u(x)/v(x) \geq 1$.

We say a random variable X or its distribution F to be heavy tailed (to the right) if $E \exp\{hX\} = \infty$ for every $h > 0$. Trivially, for F to be heavy tailed it is necessary that $\overline{F}(x) = 1 - F(x) > 0$ for all x. An important class of heavy-tailed distributions is the subexponential class. By definition, a distribution F on $[0, \infty)$ is said to be subexponential (notation $F \in \mathcal{S}$) if the relation

$$\overline{F^{*n}}(x) \sim n\overline{F}(x) \tag{1.1}$$

holds for some (or, equivalently, for all) $n = 2, 3, \ldots$; see Embrechts et al. [9]. More generally, a distribution F on $(-\infty, \infty)$ is still said to be subexponential (to the right) if the distribution $F^+(x) = F(x)1_{(x\geq 0)}$ is subexponential. Applying the Proposition of Sgibnev [26], it is easy to see that relation (1.1) remains valid for the latter general case. Thus, for n independent, identically distributed, and real-valued random variables $X_1, \ldots, X_n$ with common distribution $F \in \mathcal{S}$, with $S_n = \sum_{k=1}^n X_k$ we have

$$\Pr(S_n > x) \sim n\overline{F}(x). \tag{1.2}$$

For the case that the real-valued random variables $X_1, \ldots, X_n$ are independent but not identically distributed, one naturally expects to replace relation (1.2) by the relation

$$\Pr(S_n > x) \sim \sum_{k=1}^{n} \overline{F_k}(x). \tag{1.3}$$

Scattered discussions at this point can be found in Ng et al. [25], Asmussen et al. [3], Geluk and De Vries [14], and Foss et al. [11], among others.

We remark that the assumption of independence among the underlying random variables appears far too unrealistic in most practical situations and it considerably limits the usefulness of obtained results.

In this paper we aim at establishing relation (1.3) for the case that the random variables $X_1, \ldots, X_n$ are dependent with corresponding subexponential distributions $F_1, \ldots, F_n$. This expresses a certain insensitivity of the asymptotic tail behavior of S_n, i.e. the subexponentiality of the marginal distributions eliminates the strength of the dependence between the summands. Our results will demonstrate that the heavier the marginal tails are the more insensitive the asymptotic tail behavior of S_n is with respect to the underlying dependence structure. Closely related works in the recent literature are Geluk and Ng [13], Albrecher et al. [1], Tang [27], Kortschak and Albrecher [17], and Ko and Tang [16], among others.

However, in ruin theory there are several works showing that insensitivity of asymptotics for heavy-tailed random variables to the underlying dependence structures is not always the case; see e.g. Mikosch and Samorodnitsky [23, 24], Korshunov et al. [18], Tang and Vernic [28], and Foss et al. [11]. We shall not expand this discussion in the current paper.

In the rest of this paper, after simply reviewing heavy-tailed distributions in Section 2 we present two main results in Section 3 and prove them in Sections 4 and 5, respectively.

2 Heavy-tailed distributions

It is well known that the class $\mathcal{S}$ is closed under tail equivalence, i.e., for two distributions F and G, if $F \in \mathcal{S}$ and $\overline{F}(x) \sim \overline{G}(x)$ then $G \in \mathcal{S}$; see Teugels [29] or Lemma A3.15 of Embrechts et al. [10].

It is also well known that every subexponential distribution F is long tailed (notation $F \in \mathcal{L}$) in the sense that the relation

$$\overline{F}(x+a) \sim \overline{F}(x)$$

holds for some (or, equivalently, for all) $a \neq 0$; see Chistyakov [6] or Lemma 1.3.5(a) of Embrechts et al. [10]. One easily sees that, for every distribution $F \in \mathcal{L}$, there is some function $a(\cdot) : [0, \infty) \to [0, \infty)$ such that the following items hold simultaneously:

$$a(x) \to \infty, \qquad a(x) = o(x), \qquad \overline{F}(x \pm a(x)) \sim \overline{F}(x). \tag{2.1}$$

Closely related is the class $\mathcal{D}$ of distributions with dominatedly-varying tails, characterized by the relation

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for some (or, equivalently, for all) $0 < y < 1$. Clearly, if $F \in \mathcal{D}$ then it holds for every $y > 0$ that

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty. \tag{2.2}$$

The intersection $\mathcal{D} \cap \mathcal{L}$ forms a popular subclass of subexponential distributions; see Goldie [15] or Proposition 1.4.4(a) of Embrechts et al. [10]. In particular, it contains the famous class $\mathcal{R}$ of distributions with regularly-varying tails characterized by the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

for some $\alpha \geq 0$ and all $y > 0$.

We conclude that $\mathcal{R} \subsetneq \mathcal{D} \cap \mathcal{L} \subsetneq \mathcal{S} \subsetneq \mathcal{L}$. For more details about heavy-tailed distributions and their applications, the reader is referred to the books by Bingham et al. [5], Embrechts et al. [10], and Asmussen [2].

We list some facts below for later use.

Lemma 2.1. *Let $X_1, \ldots, X_n$ be n independent random variables with distributions $F_1, \ldots, F_n$, respectively.*

(a) If $F_1 \in \mathcal{L}$ and $F_2 \in \mathcal{L}$ then $F_1 * F_2 \in \mathcal{L}$ and $\Pr(X_1 + X_2 > x) \sim \Pr\left(X_1^+ + X_2 > x\right)$;

(b) $F_1^+ * F_2^+ \in \mathcal{S}$ if and only if $\Pr\left(X_1^+ + X_2^+ > x\right) \sim \overline{F_1}(x) + \overline{F_2}(x)$;

(c) If $F_k \in \mathcal{S}$ for all $1 \leq k \leq n$ and $F_i * F_j \in \mathcal{S}$ for all $1 \leq i \neq j \leq n$ then $F_1 * \cdots * F_n \in \mathcal{S}$ and relation (1.3) holds.

Proof. Part (a) is a combination of Lemma 1 of Geluk [12] and Lemma 4.2 of Ng et al. [25], while part (b) can be found in Embrechts and Goldie [8]. To prove part (c), note that, by part (a), the relation

$$\Pr(X_i + X_j > x) \sim \Pr\left(X_i^+ + X_j^+ > x\right)$$

holds for all $1 \le i \ne j \le n$. Hence by the assumption $F_i * F_j \in \mathcal{S}$ and closure of $\mathcal{S}$ under tail equivalence, we have $F_i^+ * F_j^+ \in \mathcal{S}$. In view of part (b), this implies that

$$\Pr\left(X_i^+ + X_j^+ > x\right) \sim \overline{F_i}(x) + \overline{F_j}(x).$$

Applying Theorem 3 of Geluk and De Vries [14], we obtain that $F_1^+ * \cdots * F_n^+ \in \mathcal{S}$ and that

$$\Pr\left(\sum_{k=1}^{n} X_k^+ > x\right) \sim \sum_{k=1}^{n} \overline{F_k}(x).$$

Relation (1.3) follows by part (a). $\square$

3 Main results

As in Section 1, let $X_1, \ldots, X_n$ be n real-valued random variables with distributions $F_1, \ldots, F_n$, respectively, and let S_n be their sum. In the first result below we consider the case $F_k \in \mathcal{D} \cap \mathcal{L}$ for all $1 \le k \le n$. As for the dependence structure between $X_1, \ldots, X_n$, we assume that:

The relation

$$\lim_{x_i \wedge x_j \to \infty} \Pr\left(|X_i| > x_i \mid X_j > x_j\right) = 0 \qquad \text{Assumption A}$$

holds for all $1 \le i \ne j \le n$.

This concept is related to what is called asymptotic independence; see e.g. Maulik and Resnick [22]. The relation in Assumption A is equivalent to the conjunction of the relations

$$\lim_{x_i \wedge x_j \to \infty} \Pr\left(X_i > x_i \mid X_j > x_j\right) = 0 \tag{3.1}$$

and

$$\lim_{x_i \wedge x_j \to \infty} \Pr\left(X_i < -x_i \mid X_j > x_j\right) = 0, \tag{3.2}$$

indicating that neither too positively nor too negatively can X_i and X_j be dependent. To see that both (3.1) and (3.2) are necessary for relation (1.3), let us look at two extreme cases below:

1. Let $X_1 = X_2 = \cdots = X$ with X distributed by $F \in \mathcal{R}$. In this case, (3.1) is violated while relation (1.2) cannot hold unless either $n = 1$ or the regularity index of F is $\alpha = 1$;

2. Let $X_1 = -X_2 = X_3 = -X_4 = \cdots = X$ with X distributed by a symmetrical distribution F with an unbounded support. In this case, (3.2) is violated while relation (1.2) cannot hold unless $n = 1$.

Despite the two extreme cases Assumption A allows a wide range of dependence structures. For example, recall that an n-dimensional distribution is called a Farlie-Gumbel-Morgenstern (FGM) distribution if it has the form

$$F(x_1, \cdots, x_n) = \prod_{k=1}^{n} F_k(x_k) \left(1 + \sum_{1 \le i < j \le n} a_{ij} \overline{F_i}(x_i) \overline{F_j}(x_j) \right), \tag{3.3}$$

where $F_1, \ldots, F_n$ are the corresponding marginal distributions and a_{ij} are real numbers fulfilling certain requirements so that $F(x_1, ..., x_n)$ is a proper n-dimensional distribution. We refer the reader to Kotz et al. [19] for a general account on multivariate FGM distributions. Clearly, if the random variables $X_1, \ldots, X_n$ follow a joint n-dimensional FGM distribution (3.3), then for all $1 \le i \ne j \le n$, the random variables X_i and X_j follow

$$F_{ij}(x_i, x_j) = F_i(x_i) F_j(x_j) \left(1 + a_{ij} \overline{F_i}(x_i) \overline{F_j}(x_j) \right),$$

so that Assumption A is satisfied.

Here comes our first main result:

Theorem 3.1. *Let $X_1, \ldots, X_n$ be n real-valued random variables with distributions $F_1, \ldots, F_n$, respectively. If $F_k \in \mathcal{D} \cap \mathcal{L}$ for all $1 \le k \le n$ and Assumption A holds then relation (1.3) holds.*

Lemma 2.1 of Davis and Resnick [7] gives a similar result but under the stronger condition that the marginal distributions $F_1, \ldots, F_n$ belong to the class $\mathcal{R}$ with tails proportionally asymptotic to each other.

The assumption $F_k \in \mathcal{D} \cap \mathcal{L}$ for all $1 \le k \le n$ in Theorem 3.1 indicates that their tails behave essentially like power functions. Hence, some important subexponential distributions such as lognormal and Weibull distributions are unfortunately excluded.

We then attempt to establish relation (1.3) for the case of subexponential marginal distributions. In doing so, however, we have to strengthen the assumption of dependence from Assumption A to the following:

There exist positive constants x_0 and c such that the inequality

$$\Pr\left(|X_i| > x_i \mid X_j = x_j \text{ with } j \in J\right) \le c\overline{F_i}(x_i) \qquad \text{Assumption B}$$

holds for all $1 \le i \le n$, $\emptyset \ne J \subset \{1, 2, \ldots, n\} \backslash \{i\}$, $x_i > x_0$, and $x_j > x_0$ with $j \in J$.

When x_j is not a possible value of X_j, i.e. $\Pr(X_j \in \Delta) = 0$ for some open set Δ containing x_j, the conditional probability in Assumption B is simply understood as 0.

This dependence structure is related to the so-called negative (or positive) regression dependence introduced by Lehmann [20]. In particular, it is easy to check that this assumption is still satisfied if the random variables $X_1, \ldots, X_n$ follow a joint n-dimensional FGM distribution (3.3) whose marginal distributions F_k for $1 \le k \le n$ are absolutely continuous and satisfy $F_k(-x) = o\left(\overline{F_k}(x)\right)$. Obviously, Assumption B implies Assumption A.

Our second main result is given below:

Theorem 3.2. *Let X_1, $\ldots$, X_n be n real-valued random variables with distributions F_1, $\ldots$, F_n, respectively. If $F_k \in \mathcal{S}$ for all $1 \le k \le n$, $F_i * F_j \in \mathcal{S}$ for all $1 \le i \ne j \le n$, and Assumption B holds, then relation (1.3) holds.*

Note that, due to the fact that the class $\mathcal{S}$ is not closed under convolution (see Leslie [21]), the condition $F_i * F_j \in \mathcal{S}$ for all $1 \le i \ne j \le n$ in Theorem 3.2 and Lemma 2.1(c) is natural.

Recently, Asmussen and Rojas-Nandayapa [4] studied the tail probability $\Pr(S_n > x)$ for a special case with dependent lognormal marginals, i.e. $X_k = e^{Y_k}$ for $1 \le k \le n$ with $(Y_1, \ldots, Y_n)^\top$ following a multivariate normal distribution with mean vector $(\mu_1, \ldots, \mu_n)^\top$ and covariance matrix $(\sigma_{ij})_{n\times n}$. Note that in this case each X_k has a tail

$$\overline{F_k}(x) = \overline{F}(x; \mu_k, \sigma_{kk}) \sim \frac{\sqrt{\sigma_{kk}}}{\sqrt{2\pi}\log x} \exp\left\{ -\frac{(\log x - \mu_k)^2}{2\sigma_{kk}} \right\}.$$

Their result is

$$\Pr(S_n > x) \sim m_n \overline{F}(x; \mu, \sigma^2), \tag{3.4}$$

where

$$\sigma^2 = \max_{1\le k\le n} \sigma_{kk}, \quad \mu = \max_{k:\sigma_{kk}=\sigma^2} \mu_k, \quad m_n = \#\{k : \sigma_{kk} = \sigma^2, \mu_k = \mu\}.$$

We remark that this result is an immediate consequence of our Theorem 3.2. Actually, Assumption B is clearly satisfied for the current case since, for every $1 \le i \le n$ and every $\emptyset \ne J \subset \{1, 2, \ldots, n\}\backslash\{i\}$, the conditional variance of Y_i given $Y_j = y_j$ for $j \in J$ is smaller than the corresponding unconditional variance. Furthermore, the right-hand side of (3.4) is asymptotically equal to $\Pr(S_n > x)$ for independent X_1, $\ldots$, X_n.

4 Proof of Theorem 3.1

In this and the next sections, for n random variables X_1, $\ldots$, X_n with distributions F_1, $\ldots$, F_n, respectively, we write $S_{n,k} = S_n - X_k = (X_1 + \cdots + X_n) - X_k$ for every $1 \le k \le n$. When $F_1 \in \mathcal{L}$, $\ldots$, $F_n \in \mathcal{L}$, the function $a(\cdot) : [0, \infty) \to [0, \infty)$ is chosen such that the items in (2.1) hold for all F_1, $\ldots$, F_n.

Lemma 4.1. *Let X_1 and X_2 be two nonnegative random variables with distributions $F_1 \in \mathcal{D} \cap \mathcal{L}$ and $F_2 \in \mathcal{D} \cap \mathcal{L}$, respectively, such that relation (3.1) holds for $(i, j) = (1, 2)$ and $(2, 1)$. Then, the distribution of the sum $X_1 + X_2$ belongs to the class $\mathcal{D} \cap \mathcal{L}$ and*

$$\Pr(X_1 + X_2 > x) \sim \overline{F_1}(x) + \overline{F_2}(x). \tag{4.1}$$

Proof. Since X_1 and X_2 are nonnegative, by (3.1) we have

$$\Pr(X_1 + X_2 > x) \ge \overline{F_1}(x) + \overline{F_2}(x) - \Pr(X_1 > x, X_2 > x) \sim \overline{F_1}(x) + \overline{F_2}(x). \tag{4.2}$$

On the other hand, recalling the function $a(\cdot)$ introduced at the beginning of this section,

$$\begin{aligned}
&\Pr\left(X_1+X_2>x\right)\\
&\leq \overline{F_1}(x-a(x))+\overline{F_2}(x-a(x))+\Pr\left(X_1>a(x),X_2>x/2\right)+\Pr\left(X_1>x/2,X_2>a(x)\right)\\
&\sim \overline{F_1}(x)+\overline{F_2}(x),
\end{aligned}$$

where we used (3.1), (2.1), and (2.2). Hence, relation (4.1) holds. The proof that the distribution of X_1+X_2 belongs to $\mathcal{D}\cap\mathcal{L}$ is straightforward. □

Lemma 4.2. *Let X_1, …, X_n be n random variables with distributions F_1, …, F_n, respectively. If Assumption A holds then for every set $\emptyset\neq I\subsetneq\{1,\ldots,n\}$ and every element $j\in\{1,\ldots,n\}\diagdown I$,*

$$\lim_{x_I\wedge x_j\to\infty}\Pr\left(\left|\sum_{i\in I}X_i\right|>x_I\ \middle|\ X_j>x_j\right)=0.$$

Proof. The result follows from the inequality

$$\Pr\left(\left|\sum_{i\in I}X_i\right|>x_I\ \middle|\ X_j>x_j\right)\leq\sum_{i\in I}\Pr\left(|X_i|>\frac{x_I}{n}\ \middle|\ X_j>x_j\right).$$

This ends the proof. □

Lemma 4.3. *Let X_1, …, X_n be n random variables with distributions $F_1\in\mathcal{L}$, …, $F_n\in\mathcal{L}$, respectively, and let Assumption A hold. Then,*

$$\Pr(S_n>x)\gtrsim\sum_{k=1}^{n}\overline{F_k}(x). \tag{4.3}$$

Proof. Recall the function $a(\cdot)$ introduced at the beginning of this section. By a standard truncation argument we have

$$\begin{aligned}
&\Pr\left(S_n>x\right)\\
&\geq\sum_{k=1}^{n}\Pr\left(S_n>x,X_k>x+a(x)\right)-\sum_{1\leq i<j\leq n}\Pr\left(X_i>x+a(x),X_j>x+a(x)\right)\\
&=P_1(x)-P_2(x).
\end{aligned}$$

Obviously, $P_2(x)=o(1)\sum_{k=1}^{n}\overline{F_k}(x)$ under Assumption A. We estimate $P_1(x)$ as

$$\begin{aligned}
P_1(x)&\geq\sum_{k=1}^{n}\Pr(S_{n,k}>-a(x),X_k>x+a(x))\\
&=\sum_{k=1}^{n}\Pr(X_k>x+a(x))-\sum_{k=1}^{n}\Pr(S_{n,k}\leq -a(x),X_k>x+a(x))\\
&\sim\sum_{k=1}^{n}\overline{F_k}(x),
\end{aligned}$$

where in the last step we used Lemma 4.2. This proves relation (4.3). □

Proof of Theorem 3.1.

Applying mathematical induction and Lemmas 4.1 and 4.2, we obtain that

$$\Pr\left(S_n > x\right) \le \Pr\left(\sum_{k=1}^{n} X_k^+ > x\right) \sim \sum_{k=1}^{n} \overline{F_k}(x).$$

By Lemma 4.3, relation (4.3) holds. This ends the proof. □

5 Proof of Theorem 3.2

Throughout this section, by saying that $(X_1^*, \dots, X_n^*)$ is an independent copy of $(X_1, \dots, X_n)$ we mean that $(X_1^*, \dots, X_n^*)$ and $(X_1, \dots, X_n)$ are two independent random vectors with the same marginal distributions and the components of $(X_1^*, \dots, X_n^*)$ are independent. As before, we write $S_{n,k}^* = S_n^* - X_k^* = (X_1^* + \cdots + X_n^*) - X_k^*$ for every $1 \le k \le n$.

Lemma 5.1. *Let $X_1, \dots, X_n$ be n random variables with distributions $F_1 \in \mathcal{L}, \dots, F_n \in \mathcal{L}$, respectively, such that Assumption B holds. Then there exist positive constants x_0 and d_n such that the inequality*

$$\Pr\left(S_{n,k} > x \mid X_k = x_k\right) \le d_n \Pr\left(S_{n,k}^* > x\right) \tag{5.1}$$

holds for all $1 \le k \le n$, $x > x_0$, and $x_k > x_0$.

Proof. We proceed the proof by induction in n. For $n = 2$, the statement follows directly from Assumption B. Assume that the statement holds for $n-1$. To prove it for n, without loss of generality we only show (5.1) for $k = n$. Clearly,

$$\begin{aligned}
&\Pr\left(S_{n-1} > x \mid X_n = x_n\right) \\
&= \Pr\left(S_{n-1} > x, \bigcup_{i=1}^{n-1}(X_i \le x_0) \,\middle|\, X_n = x_n\right) + \Pr\left(S_{n-1} > x, \bigcap_{i=1}^{n-1}(X_i > x_0) \,\middle|\, X_n = x_n\right) \\
&= Q_1(x) + Q_2(x).
\end{aligned} \tag{5.2}$$

By our inductive assumption and Lemma 2.1(a), it holds for all $x_n > x_0$ and all large x that

$$\begin{aligned}
Q_1(x) &\le \sum_{i=1}^{n-1} \Pr\left(S_{n-1,i} > x - x_0 \mid X_n = x_n\right) \\
&\le d_{n-1} \sum_{i=1}^{n-1} \Pr\left(S_{n-1,i}^* > x - x_0\right) \\
&\lesssim d_{n-1}(n-1) \Pr\left(S_{n-1}^* > x\right).
\end{aligned} \tag{5.3}$$

For $Q_2(x)$, because of Assumption B, by conditioning also on the random variables $X_2, \dots, X_{n-1}$, we have

$$Q_2(x) \le c \Pr\left(X_1^* + X_2 + \cdots + X_{n-1} > x, \bigcap_{i=2}^{n-1}(X_i > x_0) \,\middle|\, X_n = x_n\right).$$

Repeating this procedure by conditioning on X_1^*, X_3, $\ldots$, X_{n-1}, we further have

$$Q_2(x) \le c^2 \Pr\left(X_1^* + X_2^* + X_3 + \cdots + X_{n-1} > x, \bigcap_{i=3}^{n-1} (X_i > x_0) \,\middle|\, X_n = x_n \right).$$

In this way, we eventually obtain that

$$Q_2(x) \le c^{n-1} \Pr\left(S_{n-1}^* > x\right). \tag{5.4}$$

Plugging (5.3) and (5.4) into (5.2) yields (5.1) with $k = n$. This ends the proof. □

Lemma 5.2. *Let X_1, $\ldots$, X_n be n non-negative random variables with distributions F_1, $\ldots$, F_n, respectively, and let $F_k \in \mathcal{S}$ for all $1 \le k \le n$ and $F_i * F_j \in \mathcal{S}$ for all $1 \le i \ne j \le n$. Then, for every function $a(\cdot) : [0, \infty) \to [0, \infty)$ with $a(x) \to \infty$ and for every $1 \le j \le n$,*

$$\Pr\left(S_n^* > x, a(x) < X_j^* \le x\right) = o(1) \sum_{k=1}^{n} \overline{F_k}(x). \tag{5.5}$$

Proof. Note that

$$\begin{aligned}
&\Pr\left(S_n^* > x, a(x) < X_j^* \le x\right) \\
&\le \int_0^x \Pr\left(x - y < S_{n,j}^* \le x\right) \mathrm{d}F_j(y) + \Pr\left(S_{n,j}^* > x\right) \overline{F_j}(a(x)) \\
&= \Pr\left(S_n^* > x\right) - \Pr\left(S_{n,j}^* \vee X_j^* > x\right) + \Pr\left(S_{n,j}^* > x\right) \overline{F_j}(a(x)).
\end{aligned}$$

Therefore, relation (5.5) follows by using Lemma 2.1(c). □

Proof of Theorem 3.2

In view of Lemma 4.3, it suffices to establish the asymptotic relation

$$\Pr(S_n > x) \lesssim \sum_{k=1}^{n} \overline{F_k}(x). \tag{5.6}$$

Since $\Pr\left(S_n > x\right) \le \Pr\left(\sum_{k=1}^{n} X_k^+ > x\right)$, we may prove (5.6) under the assumption that each X_k is nonnegative. For all large x,

$$\begin{aligned}
\Pr\left(S_n > x\right) &\le \Pr\left(\bigcup_{k=1}^{n} (X_k > x - a(x))\right) + \Pr\left(S_n > x, \bigcap_{k=1}^{n} (X_k \le x - a(x))\right) \\
&\le \sum_{k=1}^{n} \Pr\left(X_k > x - a(x)\right) + \Pr\left(S_n > x, a(x) < \bigvee_{k=1}^{n} X_k \le x - a(x)\right) \\
&\lesssim \sum_{k=1}^{n} \overline{F_k}(x) + \sum_{k=1}^{n} \Pr(S_n > x, a(x) < X_k \le x - a(x)) \\
&= \sum_{k=1}^{n} \overline{F_k}(x) + \sum_{k=1}^{n} \int_{a(x)}^{x-a(x)} \Pr(S_{n,k} > x - y | X_k = y) \mathrm{d}F_k(y).
\end{aligned}$$

Using Lemmas 5.1 and 5.2, the last term above is bounded by

$$\begin{aligned} d_n \sum_{k=1}^{n} \int_{a(x)}^{x-a(x)} \Pr\left(S_{n,k}^* > x - y\right) \mathrm{d}F_k(y) &= d_n \sum_{k=1}^{n} \Pr\left(S_n^* > x, a(x) < X_k^* < x - a(x)\right) \\ &= o(1) \sum_{k=1}^{n} \overline{F_k}(x). \end{aligned}$$

This proves relation (5.6). □

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