

Asymptotic Behavior of Tail and Local Probabilities for Sums of Subexponential Random Variables

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Abstract

Let $\{X_k, k \geq 1\}$ be a sequence of independent and identically distributed random variables with common subexponential distribution function concentrated on $(-\infty, \infty)$, and let τ be a nonnegative and integer-valued random variable with a finite mean and which is independent of the sequence $\{X_k, k \geq 1\}$. This paper investigates asymptotic behavior of the tail probabilities $\mathbb{P}(\cdot > x)$ and the local probabilities $\mathbb{P}(x < \cdot \leq x + h)$ of the quantities $X_{(n)} = \max_{0 \leq k \leq n} X_k$, $S_n = \sum_{k=0}^n X_k$ and $S_{(n)} = \max_{0 \leq k \leq n} S_k$, $n \geq 1$, and their randomized versions $X_{(\tau)}$, S_τ and $S_{(\tau)}$, where $X_0 = 0$ by convention and $h > 0$ is arbitrarily fixed.

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1 Introduction

Throughout this paper, $\{X_k, k \geq 1\}$ denotes a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function (d.f.) $F(x) = 1 - \bar{F}(x) = \mathbb{P}(X_1 \leq x)$ which satisfies $\bar{F}(x) > 0$ for any $x \in \mathbb{R}$. To simplify notation, we may say that the X_k for $k \geq 1$ are independent copies of a generic random variable X . The mean of X , if it exists, is denoted by μ_F . With the convention $X_0 = 0$, we write, for $n \geq 1$,

$$X_{(n)} = \max_{0 \leq k \leq n} X_k, \quad S_n = \sum_{k=0}^n X_k, \quad S_{(n)} = \max_{0 \leq k \leq n} S_k.$$

Since in most cases the accurate distributions for the sum S_n and the maximum $S_{(n)}$ are not available, deriving asymptotic relationships for their tail probabilities becomes an important approach. In this, an important role is played by a well-known class of heavy-tailed distribution functions, namely the *subexponential* class \mathcal{S} . By definition, a distribution function F concentrated on $(0, \infty)$ belongs to \mathcal{S} if and only if, for each $n \geq 1$, the tail probabilities of the sum and the maximum of the first n random variables of $\{X_k, k \geq 1\}$ are asymptotically of the same order, i.e.,

$$\mathbb{P}(S_n > x) \sim \mathbb{P}(X_{(n)} > x) \quad \text{as } x \rightarrow \infty. \quad (1.1)$$

More generally, a distribution function F concentrated on $(-\infty, \infty)$ belongs to \mathcal{S} if $F^+(x) = F(x)\mathbb{I}_{(x>0)}$ does; see, for example, Foss and Zachary (2003). We refer the reader to Embrechts *et al.* (1997) for nice reviews of the class \mathcal{S} with applications to insurance and finance.

Sgibnev (1996) investigated the asymptotic behavior of the tail probability of $S_{(n)}$. Especially, when F is subexponential, his result indicates a simple asymptotic relationship

$$\mathbb{P}(S_{(n)} > x) \sim n\bar{F}(x) \quad \text{as } x \rightarrow \infty. \quad (1.2)$$

Let τ be a nonnegative and integer-valued random variable independent of the sequence $\{X_k, k \geq 1\}$. We notice that, in many cases, it is $S_{(\tau)}$, rather than $S_{(n)}$, that needs to be approximated. The first purpose of this paper is to obtain an extension of result (1.2) to the random sum $S_{(\tau)}$, with emphasis on the case where the random walk S_n has a negative drift.

The second purpose of this paper is to study the local asymptotic behavior of the random walk S_n . Suppose that there is a positive function $f(x)$ such that, for any $h > 0$,

$$\frac{1}{h}\mathbb{P}(x < X \leq x + h) \sim f(x) \quad \text{as } x \rightarrow \infty.$$

Sgibnev (1996) proved that, if $f(x)$ is regularly varying at infinity with exponent $\alpha \leq -1$ (see the definition of the class $\mathcal{R}_{-\alpha}$ in Section 2), then

$$\frac{1}{h}\mathbb{P}(x < S_{(n)} \leq x + h) \sim nf(x) \quad \text{as } x \rightarrow \infty. \quad (1.3)$$

Most recently, in studying the local behavior of the random walk maxima, Asmussen *et al.* (2002) obtained the similar local result that

$$\frac{1}{h}\mathbb{P}(x < S_n \leq x + h) \sim nf(x) \quad \text{as } x \rightarrow \infty, \quad (1.4)$$

under the assumption that $f(x)$ is the tail probability of a subexponential distribution function satisfying a certain condition (see the definition of the class \mathcal{S}^* in Section 2). Formulae (1.3) and (1.4) indicate that S_n and $S_{(n)}$ inherit the local asymptotic behavior

of the increment X . Closely related discussions can also be found in Klüppelberg (1989), Bertoin and Doney (1994), Tang (2002) and Asmussen *et al.* (2003). This paper looks at the local results (1.3) and (1.4) again, and extends them to the randomized case of S_τ and $S_{(\tau)}$.

The outline of this paper is as follows: Section 2 lists preliminaries and in particular recalls some important classes of heavy-tailed distributions; Section 3 is devoted to the tail asymptotic behavior of S_τ and $S_{(\tau)}$; Section 4 investigates the local asymptotic behavior of S_n , S_τ and $S_{(\tau)}$.

2 Preliminaries

In the sequel, all limit relationships are as $x \rightarrow \infty$ unless otherwise stated. For two positive infinitesimals $A(x)$ and $B(x)$, we write $A(x) \asymp B(x)$ if $0 < \liminf A(x)/B(x) \leq \limsup A(x)/B(x) < \infty$. We further write $A(x) \sim B(x)$ if both the limits are equal to 1.

Following a number of recent researchers in applied probability and risk theory, we restrict our interests to heavy-tailed distributions. A random variable X (or its distribution function F) is said to be *heavy tailed* on the right if $\mathbb{E}e^{rX} = \infty$ for any $r > 0$. In addition to the class \mathcal{S} , there are some other important classes of heavy-tailed distributions which are intimately related to the class \mathcal{S} . A distribution function F is said to belong to the class \mathcal{L} of *long-tailed* distribution functions if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$$

for any $y > 0$ (or, equivalently, for some $y > 0$); F is said to belong to the class \mathcal{D} of distribution functions with *dominatedly varying* tails if

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for any $y \in (0, 1)$ (or, equivalently, for some $y \in (0, 1)$). Another class of heavy-tailed distribution functions is \mathcal{S}^* , which was introduced by Klüppelberg (1988). By definition, a distribution function F is said to belong to the class \mathcal{S}^* if and only if $m = \int_0^\infty \overline{F}(t)dt$ is finite and

$$\lim_{x \rightarrow \infty} \int_0^x \frac{\overline{F}(x-t)}{\overline{F}(x)} \overline{F}(t)dt = 2m. \quad (2.1)$$

The class \mathcal{S}^* and the intersection $\mathcal{D} \cap \mathcal{L}$ are two well-known subclasses of subexponential distribution functions. A useful subclass of $\mathcal{D} \cap \mathcal{L}$ is the class *ERV* of distribution functions with *extended regularly varying* tails. This class was introduced by Bingham *et al.* (1987) and applied more recently to precise large deviations. By definition, a distribution function F is said to belong to the class *ERV* if there exist some α, β with $0 \leq \alpha \leq \beta < \infty$ and

such that

$$y^{-\beta} \leq \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha} \quad \text{for all } y > 1. \quad (2.2)$$

Sometimes we write the regularity property in (2.2) as $F \in ERV(-\alpha, -\beta)$. When $\alpha = \beta \geq 0$ relation (2.2) describes the class $\mathcal{R}_{-\alpha}$ of distribution functions with *regularly varying* tails. It is well known that, for any α, β and γ with $0 \leq \alpha \leq \gamma \leq \beta$,

$$\mathcal{R}_{-\gamma} \subset ERV(-\alpha, -\beta) \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L} \quad (2.3)$$

and that, if the distribution function F has a finite mean, then

$$F \in \mathcal{D} \cap \mathcal{L} \implies F \in \mathcal{S}^* \subset \mathcal{S}. \quad (2.4)$$

The proof of (2.3) can be found in Bingham *et al.* (1987, Chapter 2), Goldie (1978) and Embrechts *et al.* (1997) and the references therein; the proof of (2.4) was given by Klüppelberg (1988).

Recall that $\{X_k, k \geq 1\}$ denotes a sequence of i.i.d. random variables with distribution function F . We present here some inequalities for the tail probabilities of S_n and $S_{(n)}$ for use later. A classical inequality in the literature says that, if $F \in \mathcal{S}$, then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\mathbb{P}(S_n > x) \leq C_\varepsilon (1 + \varepsilon)^n \overline{F}(x) \quad (2.5)$$

holds for all $n \in \mathbb{N}$ and all $x \geq 0$. We remark that, in the original work by Chistyakov (1964) and Athreya and Ney (1972), the distribution function F is assumed to be concentrated on $(0, \infty)$, but the extension of this result to the general case of F concentrated on $(-\infty, \infty)$ is trivial. Noticing that

$$\mathbb{P}(S_{(n)} > x) \leq \sum_{k=1}^n \mathbb{P}(S_k > x),$$

we can also derive from (2.5) that, if $F \in \mathcal{S}$, then, for any $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that

$$\mathbb{P}(S_{(n)} > x) \leq D_\varepsilon (1 + \varepsilon)^n \overline{F}(x) \quad (2.6)$$

for all $n \in \mathbb{N}$ and all $x \geq 0$. We remark that the bounds given in (2.5) and (2.6) are too rough with respect to n . Fortunately, some more precise bounds are available in the literature. Tang and Yan (2002) proved that, if F is in \mathcal{D} and has a finite mean μ , then, for any $\gamma > \mu^+ = \max\{\mu, 0\}$, there exists a constant $E_\gamma > 0$ such that

$$\mathbb{P}(S_n > x) \leq E_\gamma n \overline{F}(x) \quad (2.7)$$

for all $n \in \mathbb{N}$ and $x \geq \gamma n$. The next inequality is from Petrov (1975): for any $0 < q < 1$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, it holds that

$$\mathbb{P}(S_{(n)} \geq x) \leq \frac{1}{q} \mathbb{P}\left(S_n \geq x - \frac{n\mathbb{E}|X|}{1-q}\right). \quad (2.8)$$

3 Asymptotic Behavior of the Tail Probabilities

Let F be a distribution function with $m = \int_0^\infty \bar{F}(t)dt$ finite. We denote its equilibrium distribution function by

$$F_e(x) := \frac{1}{m} \int_0^x \bar{F}(t)dt, \quad x > 0.$$

Clearly, F_e is an absolutely continuous distribution function concentrated on $(0, \infty)$ and has a hazard rate function

$$r_e(x) := -\frac{d}{dx} \ln \bar{F}_e(x) = \frac{\bar{F}(x)}{\int_x^\infty \bar{F}(t)dt}, \quad x > 0,$$

where the latter equality holds almost everywhere with respect to the Lebesgue measure. It is easy to prove that, if $F \in \mathcal{D}$, then $r_e(x) = O(x^{-1})$; see also the proof of (3.10) below.

Theorem 3.1. *Let τ have a finite mean and be independent of the sequence $\{X_k, k \geq 1\}$. Then*

$$\mathbb{P}(S_{(\tau)} > x) \sim \mathbb{P}(S_\tau > x) \sim \mathbb{P}(X_{(\tau)} > x) \sim \mathbb{E}\tau \bar{F}(x) \quad (3.1)$$

if one of the following conditions holds:

- (i) $F \in \mathcal{S}$ and $\mathbb{E} \exp\{\delta\tau\} < \infty$ for some $\delta > 0$;
- (ii) $F \in \mathcal{D} \cap \mathcal{L}$, $\mu_F < 0$ and $\mathbb{P}(\tau > x) = o(r_e(x))$;
- (iii) $F \in ERV(-\alpha, -\beta)$ for some α, β with $1 < \alpha \leq \beta < \infty$ and $\mu_F < 0$.

Compared with (1.1), the asymptotic relation (3.1) describes some natural properties of subexponential distributions. The sufficient condition (ii) of Theorem 3.1 weakens the conditions of Theorem 2.3 in Ng *et al.* (2002). We refer the reader to Foss and Zachary (2003) for closely related discussions in a more general probabilistic model.

Proof. The proof of the asymptotic relationship

$$\mathbb{P}(X_{(\tau)} > x) \sim \mathbb{E}\tau \bar{F}(x) \quad (3.2)$$

is direct. In fact, for any fixed $n \in \mathbb{N}$ we have

$$\mathbb{P}(X_{(n)} > x) \sim n\bar{F}(x) \quad \text{and} \quad \mathbb{P}(X_{(n)} > x) \leq n\bar{F}(x) \quad \text{for any } x \geq 0.$$

Hence (3.2) follows immediately since $\mathbb{E}\tau < \infty$.

In order to prove the remaining part of (3.1), we consider it as a conjunction of two asymptotic relationships:

$$\mathbb{P}(S_{(\tau)} > x) \sim \mathbb{E}\tau \bar{F}(x) \quad \text{and} \quad \mathbb{P}(S_\tau > x) \sim \mathbb{E}\tau \bar{F}(x). \quad (3.3)$$

Trivially,

$$\mathbb{P}(S_{(\tau)} > x) = \sum_{n=1}^{\infty} \mathbb{P}(S_{(n)} > x) \mathbb{P}(\tau = n). \quad (3.4)$$

We prove (3.3) for each of the conditions (i), (ii), (iii) separately.

(i) $F \in \mathcal{S}$ and $\mathbb{E} \exp\{\delta\tau\} < \infty$ for some $\delta > 0$. For this case, the second relation in (3.3) is well known; see, for example, Chover *et al.* (1973). Now, from (3.4) and (1.2),

$$\mathbb{P}(S_{(\tau)} > x) \sim \sum_{n=1}^{\infty} n\bar{F}(x) \mathbb{P}(\tau = n) = \mathbb{E}\tau\bar{F}(x),$$

where the asymptotic relationship is guaranteed by the dominated convergence theorem, which is justified by (2.6) and the finiteness of $\mathbb{E} \exp\{\delta\tau\}$. This gives the first relation in (3.3).

(ii) $F \in \mathcal{D} \cap \mathcal{L}$, $\mu_F < 0$ and $\mathbb{P}(\tau > x) = o(r_e(x))$. For a fixed $q \in (0, 1)$, we choose a constant γ such that

$$\gamma > \mathbb{E}X^+ + \frac{\mathbb{E}|X|}{1-q} + 1,$$

where $X^+ = \max\{X, 0\}$. Relying on this γ , we divide the series in (3.4) into two parts:

$$\mathbb{P}(S_{(\tau)} > x) = \left(\sum_{1 \leq n < x/\gamma} + \sum_{n \geq x/\gamma} \right) \mathbb{P}(S_{(n)} > x) \mathbb{P}(\tau = n) =: I_1 + I_2. \quad (3.5)$$

Clearly, when $1 \leq n < x/\gamma$,

$$x - \frac{n\mathbb{E}|X|}{1-q} \geq \gamma n - \frac{n\mathbb{E}|X|}{1-q} \geq (\mathbb{E}X^+ + 1)n = \tilde{\gamma}n.$$

Let $X_k^+ = \max\{X_k, 0\}$ for $k \geq 1$. We then obtain by (2.8) and (2.7) that

$$\begin{aligned} \mathbb{P}(S_{(n)} > x) &\leq \frac{1}{q} \mathbb{P}\left(\sum_{k=1}^n X_k^+ \geq x - \frac{n\mathbb{E}|X|}{1-q}\right) \\ &\leq \frac{E_{\tilde{\gamma}}}{q} n \mathbb{P}\left(X^+ \geq x - \frac{n\mathbb{E}|X|}{1-q}\right) \\ &\leq \frac{E_{\tilde{\gamma}}}{q} n \mathbb{P}\left(X^+ \geq x \left(1 - \frac{\mathbb{E}|X|}{\gamma(1-q)}\right)\right) \\ &\asymp n\bar{F}(x), \end{aligned}$$

where, in the last step we used the fact that, if $F \in \mathcal{D}$, then

$$\bar{F}(xy) \asymp \bar{F}(x) \quad \text{for any } y > 0. \quad (3.6)$$

So by the dominated convergence theorem and (1.2) we conclude

$$I_1 \sim \sum_{1 \leq n < x/\gamma} n\bar{F}(x) \mathbb{P}(\tau = n) \sim \mathbb{E}\tau\bar{F}(x). \quad (3.7)$$

As is well known, if $F_e \in \mathcal{S}$, then

$$\mathbb{P}\left(\sup_{0 \leq n < \infty} S_n > x\right) \sim \frac{1}{-\mu_F} \int_x^\infty \bar{F}(t) dt; \quad (3.8)$$

see Veraverbeke (1977) and Embrechts and Veraverbeke (1982). We remark that the condition $F_e \in \mathcal{S}$ can be satisfied if F is in \mathcal{D} and has a finite mean; see Embrechts and Omey (1984). From (3.8) and (3.6),

$$\begin{aligned} I_2 &\leq \mathbb{P}\left(\sup_{0 \leq n < \infty} S_n > x\right) \sum_{n \geq x/\gamma} \mathbb{P}(\tau = n) \\ &\sim \frac{1}{-\mu_F} \int_x^\infty \bar{F}(t) dt \mathbb{P}(\tau \geq x/\gamma) \\ &\leq \frac{1}{-\mu_F} \frac{\int_{x/\gamma}^\infty \bar{F}(t) dt}{\bar{F}(x/\gamma)} \mathbb{P}(\tau \geq x/\gamma) \frac{\bar{F}(x/\gamma)}{\bar{F}(x)} \bar{F}(x) \\ &\asymp \frac{\mathbb{P}(\tau \geq x/\gamma)}{r_e(x/\gamma)} \bar{F}(x) \\ &= o(\bar{F}(x)). \end{aligned} \quad (3.9)$$

Substituting (3.7) and (3.9) into (3.5), we obtain the first asymptotic relationship in (3.3). Analogously to the proof of (3.7), we have

$$\begin{aligned} \mathbb{P}(S_\tau > x) &\geq \mathbb{P}(S_\tau > x) \\ &\geq \sum_{1 \leq n < x/\gamma} \mathbb{P}(S_n > x) \mathbb{P}(\tau = n) \\ &\sim \sum_{1 \leq n < x/\gamma} n \bar{F}(x) \mathbb{P}(\tau = n) \\ &= \mathbb{E}\tau \bar{F}(x). \end{aligned}$$

This, coupled with the first asymptotic relationship in (3.3), implies the second asymptotic relationship in (3.3).

(iii) $F \in ERV(-\alpha, -\beta)$ for some α, β with $1 < \alpha \leq \beta < \infty$ and $\mu_F < 0$. First we recall (2.3) and the second part of Theorem 3.1. In order to prove (3.1) for the present case, it suffices to prove $\mathbb{P}(\tau > x) = o(r_e(x))$. In view of the finiteness of $\mathbb{E}\tau$, we only need to prove $r_e(x) \asymp 1/x$, i.e.

$$0 < \liminf_{x \rightarrow \infty} \frac{x \bar{F}(x)}{\int_x^\infty \bar{F}(t) dt} \leq \limsup_{x \rightarrow \infty} \frac{x \bar{F}(x)}{\int_x^\infty \bar{F}(t) dt} < \infty. \quad (3.10)$$

In fact, for any ε such that $0 < \varepsilon < 2^{\alpha-1} - 1$, from the definition of the class ERV with y in (2.2) set equal to 2, we know that, for all sufficiently large $x > 0$,

$$\bar{F}(2x) \leq (1 + \varepsilon) 2^{-\alpha} \bar{F}(x).$$

It follows that

$$\begin{aligned}
\int_x^\infty \bar{F}(y) dy &= \sum_{k=0}^{\infty} \int_{2^k x}^{2^{k+1} x} \bar{F}(t) dt \\
&\leq \sum_{k=0}^{\infty} 2^k x \bar{F}(2^k x) \\
&\leq \sum_{k=0}^{\infty} 2^k x ((1 + \varepsilon) 2^{-\alpha})^k \bar{F}(x) \\
&= C x \bar{F}(x),
\end{aligned}$$

where $C := \sum_{k=0}^{\infty} 2^k ((1 + \varepsilon) 2^{-\alpha})^k$ is a finite constant. Thus, for all sufficiently large $x > 0$,

$$0 < \frac{1}{C} \leq \frac{x \bar{F}(x)}{\int_x^\infty \bar{F}(y) dy} \leq \frac{x \bar{F}(x)}{\int_x^{2x} \bar{F}(y) dy} \leq \frac{\bar{F}(x)}{\bar{F}(2x)} \leq (1 + \varepsilon) 2^\beta < \infty,$$

and therefore (3.10) holds.

This ends the proof of Theorem 3.1. \square

From the proof of Theorem 3.1, we can see that the restriction $F \in ERV$ in the sufficient condition (iii) of Theorem 3.1 can be slightly weakened as follows.

Corollary 3.2. *Relation (3.1) holds if the following two conditions hold simultaneously:*

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = 1 \quad (3.11)$$

and

$$\limsup_{x \rightarrow \infty} \frac{\bar{F}(xy_0)}{\bar{F}(x)} \leq y_0^{-\alpha} \quad \text{for some } y_0 > 1 \text{ and } \alpha > 1. \quad (3.12)$$

We remark that relation (3.11) describes a kind of regularity of the tail probability \bar{F} . It was introduced by Cline (1994), who called it ‘*intermediate regular varying*’ property. It has been applied by many researchers in the recent literature.

Proof. Considering the proof of the third part of Theorem 3.1, we easily see that conditions (3.11) and (3.12) imply (3.10); therefore $r_e(x) \asymp x^{-1}$ and $\mathbb{P}(\tau > x) = o(r_e(x))$. In addition, (3.11) implies $F \in \mathcal{D} \cap \mathcal{L}$. Thus, by the second part of Theorem 3.1, we know that (3.1) holds in the present case. \square

4 Asymptotic Behavior of the Local Probabilities

For notational convenience, we introduce

$$\mathbb{Q}_h(X; x) = \mathbb{Q}_h(F; x) = \frac{1}{h} (\bar{F}(x) - \bar{F}(x+h)) = \frac{1}{h} F(x, x+h], \quad h > 0.$$

This section is devoted to the local asymptotic behavior of the sums S_n and S_τ and the maxima $S_{(n)}$ and $S_{(\tau)}$. Hereafter, the distribution function F is always assumed to satisfy that

$$\mathbb{Q}_h(F; x) \sim Cf(x) \quad \text{for some } C > 0 \text{ and any } h > 0, \quad (4.1)$$

where the function $f(x): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is proportional to the tail of some distribution function from the class \mathcal{S}^* . Without loss of generality, we assume that $f(x)$ is nonincreasing in $x \in (0, \infty)$.

Asmussen *et al.* (2002, Lemma 1) obtained that, if F_1 and F_2 concentrated on $(0, \infty)$ satisfy condition (4.1) with the coefficients $C_1, C_2 > 0$, then, for any $h > 0$,

$$\mathbb{Q}_h(F_1 * F_2; x) \sim (C_1 + C_2)f(x). \quad (4.2)$$

This indicates that the operator $\mathbb{Q}_h(F; x)$ is additive in F in the asymptotic sense. The extension of (4.2) to the general case where F_i are concentrated on $(-\infty, \infty)$ is immediate. Actually, we introduce two independent random variables Y_1 and Y_2 , which are distributed by F_1 and F_2 , respectively. Then we divide $\mathbb{Q}_h(F_1 * F_2; x) = h^{-1}\mathbb{P}(x < Y_1 + Y_2 \leq x + h) = J_1 + J_2 + J_3$ into three parts J_1, J_2, J_3 according to $(Y_1 > 0, Y_2 > 0)$, $(Y_1 > 0, Y_2 \leq 0)$ and $(Y_1 \leq 0, Y_2 > 0)$, respectively. It is not difficult to check that

$$J_1 \sim \overline{F_1}(0)\overline{F_2}(0) \left(\frac{C_1}{\overline{F_1}(0)} + \frac{C_2}{\overline{F_2}(0)} \right) f(x), \quad J_2 \sim C_1 f(x)F_2(0), \quad J_3 \sim C_2 f(x)F_1(0).$$

Therefore (4.2) follows immediately.

By induction, we can derive from (4.2) that $\mathbb{Q}_h(S_n; x) \sim n\mathbb{Q}_h(X; x)$ for each $n \geq 1$. On the other hand, starting from (4.2) and following the same line as the proof of the theorem of Sgibnev (1996), we can also obtain that $\mathbb{Q}_h(S_{(n)}; x) \sim n\mathbb{Q}_h(X; x)$ for $n \geq 1$. Furthermore, the proof that $\mathbb{Q}_h(X_{(n)}; x) \sim n\mathbb{Q}_h(X; x)$ for $n \geq 1$ is straightforward since

$$\mathbb{Q}_h(X_{(n)}; x) = \mathbb{Q}_h(X; x) \sum_{k=0}^{n-1} F^k(x+h)F^{n-k-1}(x).$$

We summarize this as follows.

Theorem 4.1. *If the distribution function F satisfies (4.1), then, for any $n \in \mathbb{N}$ and $h > 0$,*

$$\mathbb{Q}_h(S_{(n)}; x) \sim \mathbb{Q}_h(S_n; x) \sim \mathbb{Q}_h(X_{(n)}; x) \sim n\mathbb{Q}_h(X; x). \quad (4.3)$$

Now we aim to extend Theorem 4.1 to the randomized case. By copying the proof of Lemma 2 in Asmussen *et al.* (2002) with some adjustments, we can obtain that, if F concentrated on $(-\infty, \infty)$ satisfies condition (4.1), then, for any $\varepsilon > 0$, there exists some $C(\varepsilon, h) > 0$ such that

$$\mathbb{Q}_h(F^{*n}; x) \leq C(\varepsilon, h)(1 + \varepsilon)^n f(x) \quad (4.4)$$

for all $n \in \mathbb{N}$ and $x \geq 0$. Since $\mathbb{Q}_h(S_{(n)}; x) \leq \sum_{k=1}^n \mathbb{Q}_h(S_k; x)$, we immediately obtain from (4.4) that, for any $\varepsilon > 0$, there exists some $D(\varepsilon, h) > 0$ such that

$$\mathbb{Q}_h(S_{(n)}; x) \leq D(\varepsilon, h)(1 + \varepsilon)^n f(x) \quad (4.5)$$

for all $n \in \mathbb{N}$ and all $x \geq 0$. Hence, if we assume $\mathbb{E} \exp\{\delta\tau\} < \infty$ for some $\delta > 0$, then by the dominated convergence theorem, it follows from (4.3) that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{Q}_h(S_\tau; x)}{\mathbb{Q}_h(X; x)} = \sum_{n=1}^{\infty} \left(\lim_{x \rightarrow \infty} \frac{\mathbb{Q}_h(S_n; x)}{\mathbb{Q}_h(X; x)} \right) \mathbb{P}(\tau = n) = \mathbb{E}\tau.$$

This proves $\mathbb{Q}_h(S_\tau; x) \sim \mathbb{E}\tau \mathbb{Q}_h(X; x)$. We can obtain $\mathbb{Q}_h(S_{(\tau)}; x) \sim \mathbb{E}\tau \mathbb{Q}_h(X; x)$ similarly. Finally, the proof for the relationship $\mathbb{Q}_h(X_{(\tau)}; x) \sim \mathbb{E}\tau \mathbb{Q}_h(X; x)$ is also immediate.

We summarize this as follows.

Theorem 4.2. *If the distribution function F satisfies (4.1) and $\mathbb{E} \exp\{\delta\tau\} < \infty$ for some $\delta > 0$, then, for any $h > 0$,*

$$\mathbb{Q}_h(S_{(\tau)}; x) \sim \mathbb{Q}_h(S_\tau; x) \sim \mathbb{Q}_h(X_{(\tau)}; x) \sim \mathbb{E}\tau \mathbb{Q}_h(X; x). \quad (4.6)$$

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