Precise large deviations for the prospective-loss process

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Abstract

In this paper, we propose a customer-arrival-based insurance risk model, in which customer's potential claims are described as i.i.d. heavy-tailed r.v.'s and premiums are the same for each policy. We obtain some precise large deviation results for the prospective loss process under a mild assumption on the random index (in our case, the customer arrival process), which is much weaker than that in the literature.

Key Words: Insurance Risk Model, Point Process, Precise Large Deviation, Subexponentiality.

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1 Introduction

In the classical insurance risk model (see Embrechts *et al.* (1997) and Rolski *et al.* (1999)), the surplus is described as the initial surplus plus the premium income with the claims taken off. A compound process is often used to model this surplus process, in which the claim number process is a counting process, and the claim sizes are assumed to form a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s). In the classical model, the premium rate is assumed to be a constant. Recently, there have

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been some works which extend the classical model by assuming that the premium is also random. However, by doing so, they encountered the problem of dependence between the premium process and the claim arrival process, and the problem becomes difficult.

In this paper we propose a different way to model the surplus of an insurance company. Rather than counting the number of claims, we count the number of customers. The model can be described as follows:

- The individual customer arrival process is $N(t) = \max\{k \geq 0 : \sigma_k \leq t\}$, where σ_k is the time the kth customer arrives, $\sigma_0 = 0$. Naturally, we assume $N(t)$ is a non-negative and integer-valued stochastic process with a mean function $\lambda(t) < \infty$ for any $t > 0$ but $\lambda(t) \to \infty$ as $t \to \infty$. In our setup, $N(t)$ can be a Poisson process, Cox process or a marked point process. We do not exclude the possibility of more than one customer coming at the same time;

- At the time σ_k, the kth customer buys an insurance contract, $k \geq 1$. The insurance company will therefore bear a risk from this policy holder within a fixed term, say τ. The term τ can be one month, one year, or infinite;

- Assume that the total potential claims due to the kth customer within the term τ is X_k, and that $\{X_k, k \geq 1\}$ forms a sequence of i.i.d. non-negative r.v.'s with a common distribution function (d.f.) F and a finite mean μ. This assumption means that we are considering all the customers coming from a certain category of the population. All the individuals in this population are indifferent in terms of the total potential claims. The price of each policy is $(1+\delta)\mu$, where the positive constant δ can be interpreted as the safety loading coefficient. Clearly, the insurance company's total net risk coming from the kth policy holder is

$$X_k - (1+\delta)\mu; \tag{1.1}$$

- We assume that the interest rate is zero. Therefore, the surplus process of the company within the period $[0, t]$ can be written as $U(t) = x - W(t)$, where x denotes the initial reserve of the company, and $W(t)$ denotes the prospective loss process

$$W(t) = \sum_{k=1}^{N(t)} \left(X_k - (1+\delta)\mu\right), \quad t \geq 0, \tag{1.2}$$

  which is, of course, a very important object in risk management.

In this paper, we address the large deviation problems for the prospective loss process $W(t)$. For some fundamental results on precise large deviations and their applications in insurance and finance, we refer the reader to Cline and Hsing (1991), Asmussen and Klüppelberg (1996), Klüppelberg and Mikosch (1997), Embrechts *et al.* (1997) (Sections 8.6 and 8.7), Mikosch and A.V. Nagaev (1998) and Tang *et al.* (2001), among others.

This paper is organized as follows: Section 2 presents some mathematical concepts and notations, recalls some key results on precise large deviations, and states our main results. The proofs of the main results are presented in Section 3.

2 Notations and Main Results

Given a non-negative r.v. X with a finite mean μ, its d.f. is denoted by $F(x) = \mathbb{P}(X \leq x)$ and its tail by $\overline{F} = 1 - F$. We say X (or its d.f. F) is heavy-tailed if it has no exponential moments. See, for example, Embrechts *et al.* (1997) for the definitions of many types of heavy-tailed subclasses and their applications to insurance and finance. The most well-known subclass of heavy-tailed distributions is the subexponential class, denoted by $\mathcal{S}$. By definition, a d.f. F with support $[0, \infty)$ belongs to $\mathcal{S}$ if for some $n \geq 2$ (or equivalently for any $n \geq 2$) it holds that

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n. \tag{2.1}$$

Recall that (see for example Lemma 1.3.5 in Embrechts *et al.* (1997)), for $F \in \mathcal{S}$,

$$\lim_{x\to\infty} \frac{\overline{F}(x+L)}{\overline{F}(x)} = 1 \qquad for\ any\ fixed\ L > 0. \tag{2.2}$$

Another well-known heavy-tailed subclass is the class $\mathcal{R}$, which consists of d.f.'s with regularly varying tails. A natural extended version of $\mathcal{R}$ is the so-called class ERV (Extended Regularly Varying class). By definition, a d.f. F with support $[0, \infty)$ belongs to ERV if

$$y^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha} \qquad for\ any\ y > 1, \tag{2.3}$$

for some $1 < \alpha \leq \beta < \infty$ (write $F \in \text{ERV}(-\alpha, -\beta)$ for simplicity). See Bingham *et al.* (1987) (p.61-76), Cline and Hsing (1991) and references therein for details of the introduction and deeper investigations of the class ERV.

Henceforth, $\{X_k, k \geq 1\}$ denotes a sequence of i.i.d., non-negative r.v.'s with a generic r.v. X and a generic d.f. F belonging to $\text{ERV}(-\alpha, -\beta)$, and $\{N(t), t \geq 0\}$ denotes a non-negative, integer-valued process, which is independent of the sequence $\{X_k, k \geq 1\}$.

We assume that $\lambda(t) = \mathbb{E}N(t) < \infty$ for all $t \geq 0$ but $\lambda(t) \to \infty$. All limit relations, unless otherwise stated, are for $t \to \infty$ (or, consequently, for $\lambda(t) \to \infty$). We denote the partial sum of $\{X_k, k \geq 1\}$ by $S_n = \sum_{k=1}^{n} X_k$, as usual.

The following result, so-called precise large deviation of S_n, is an easy consequence of a very general result obtained in Cline and Hsing (1991) (see also Klüppelberg and Mikosch (1997)): for any fixed $\gamma > 0$, it holds that

$$\mathbb{P}\left(\sum_{k=1}^{n}(X_k - \mu) > x\right) \sim n\overline{F}(x) \quad \textit{uniformly for } x \geq \gamma n, \quad n \to \infty. \tag{2.4}$$

Recently, Klüppelberg and Mikosch (1997) extended (2.4) to the case of random sums. They showed that

$$\mathbb{P}\left(\sum_{k=1}^{N(t)} X_k - \mathbb{E}\sum_{k=1}^{N(t)} X_k > x\right) \sim \lambda(t)\overline{F}(x) \quad \textit{uniformly for } x \geq \gamma\lambda(t), \tag{2.5}$$

under the condition that

$$N(t)/\lambda(t) \to_p 1, \qquad \textit{Assumption A}$$

and

$$\sum_{n > (1+\theta)\lambda(t)} \mathbb{P}\left(N(t) = n\right)(1+\varepsilon)^n = o(1) \qquad \textit{Assumption B}$$

for any constant $\theta > 0$ and some small $\varepsilon > 0$.

Clearly, Assumption B is a quite strong condition. Even the commonly used renewal counting process may fail to satisfy this condition. Recently, Tang *et al.* (2001) (Theorem 2.1) weakened Assumption B to the condition that, for some given $\beta > 1$,

$$\mathbb{E}N(t)^{\beta+\varepsilon}\mathbb{I}_{(N(t)>(1+\theta)\lambda(t))} = O(\lambda(t)) \qquad \textit{Assumption } C_\beta$$

for any fixed small constant $\theta > 0$ and some small $\varepsilon > 0$. They also proved that Assumption C_β can be applied at least to the so-called compound renewal process.

The main purpose of the present paper is to investigate the precise large deviations of the prospective loss process $W(t)$ of our insurance risk model. First of all, we state a precise large deviation result for the non-random case as follows.

Theorem 2.1. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d.* ERV$(-\alpha, -\beta)$ *r.v.'s with $1 < \alpha \leq \beta < \infty$. Then, for every fixed $\gamma > 0$, it holds that, uniformly for $x \geq \gamma n$*

$$\mathbb{P}\left(\sum_{k=1}^{n}(X_k - (1+\delta)\mu) > x\right) \sim n\overline{F}(x + \delta n\mu), \quad n \to \infty, \tag{2.6}$$

and that, uniformly for $x \gg n$,

$$\mathbb{P}\left(\sum_{k=1}^{n}(X_k - (1+\delta)\mu) > x\right) \sim n\overline{F}(x), \quad n \to \infty. \tag{2.7}$$

Here and henceforth, we write $x \gg n$ to mean that the related asymptotic result holds uniformly for $x \geq f(n)n$ for any choice of sequence $f(n)$ such that $0 \leq f(n) \to \infty$ as $n \to \infty$. The notation $x \gg \lambda(t)$ can be understood in a similar way.

Proof. The result (2.6) is a direct consequence of (2.4). Furthermore, from the definition of (2.3), we can obtain that

$$\overline{F}(x + o(x)) \sim \overline{F}(x), \quad x \to \infty, \tag{2.8}$$

for any $o(x)$, which implies that $\overline{F}(x+\delta n\mu) \sim \overline{F}(x)$ holds uniformly for $x \gg n$ as $n \to \infty$. This proves the assertion (2.7). □

Now we are processing to state the main results of the paper. The following result extends the precise large deviation results (2.6) and (2.7) to the random sums $W(t)$ in (1.2) under Assumptions A on $N(t)$.

Theorem 2.2. *Let* $\{X_k, k \geq 1\}$ *be a sequence of i.i.d.* ERV$(-\alpha, -\beta)$ *r.v.'s with* $1 < \alpha \leq \beta < \infty$. *We assume that* $\{X_k, k \geq 1\}$ *is independent of the non-negative, integer-valued process* $\{N(t), t \geq 0\}$. *Furthermore, we suppose that* $N(t)$ *satisfies Assumption A. Then for any fixed* $\gamma > 0$ *it holds that*

$$\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k - (1+\delta)\mu) > x\right) \sim \lambda(t)\overline{F}(x + \delta\lambda(t)\mu) \tag{2.9}$$

uniformly for $x \geq \gamma\lambda(t)$.

By (2.8), we obtain a corollary of Theorem 2.2:

Corollary 2.3. *Under the conditions of Theorem 2.2, we have that*

$$\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k - (1+\delta)\mu) > x\right) \sim \lambda(t)\overline{F}(x) \tag{2.10}$$

holds uniformly for $x \gg \lambda(t)$.

Note that the way of centering the collective risks in (2.9) and (2.10) differs from that in (2.5). One naturally raises a question: *how big is the difference between the asymptotics*

(2.5) and (2.9) (or (2.10))? Our answer is: *under Assumption C_β, they are equivalent to each other.* In fact, we obtain a little more general result:

Theorem 2.4. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d.* ERV$(-\alpha, -\beta)$ *r.v.'s with $1 < \alpha \leq \beta < \infty$. We assume that $\{X_k, k \geq 1\}$ is independent of the non-negative, integer-valued process $\{N(t), t \geq 0\}$. Then, under Assumption C_β, for any fixed $\gamma > 0$ it holds that*

$$\mathbb{P}\left(\sum_{k=1}^{N(t)} X_k - \mathbb{E}\sum_{k=1}^{N(t)} X_k > x\right) \sim \mathbb{P}\left(\sum_{k=1}^{N(t)} (X_k - \mu) > x\right) \sim \lambda(t)\overline{F}(x) \tag{2.11}$$

uniformly for $x \geq \gamma\lambda(t)$. If we further restrict the x-region to $x \gg \lambda(t)$, then, uniformly,

$$\mathbb{P}\left(\sum_{k=1}^{N(t)} X_k - \mathbb{E}\sum_{k=1}^{N(t)} X_k > x\right) \sim \mathbb{P}\left(\sum_{k=1}^{N(t)} (X_k - (1+\delta)\mu) > x\right) \sim \lambda(t)\overline{F}(x). \tag{2.12}$$

The solvency is one of main concerns in insurance companies. We want to be sure that prospective loss at any moment t within interval $[0, T]$ does not exceed x. Therefore we want to find "solvency" probability (also known as ruin probability in a finite time horizon) $\mathbb{P}\left(\sup_{0\leq t\leq T} W(t) > x\right)$. The following result gives an asymptotic relation between the "solvency" probability and the tail probability of the potential claim of each customer.

Theorem 2.5. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. subexponential r.v.'s, which is independent of the non-negative, integer-valued process $\{N(t), t \geq 0\}$. We assume that, for a given $T > 0$,*

$$\mathbb{E}\,(1+\varepsilon)^{N(T)} < \infty \quad for\ some \quad \varepsilon > 0. \qquad \textit{Assumption D}$$

Then, it holds that

$$\mathbb{P}\left(\sup_{0\leq t\leq T} \sum_{k=1}^{N(t)} (X_k - (1+\delta)\mu) > x\right) \sim \lambda(T)\overline{F}(x), \qquad x \to \infty. \tag{2.13}$$

It is easy to see that Assumption B implies Assumption D, and the homogeneous Poisson arrival process satisfies Assumption D. We also remark that the asymptotics (2.13) is a rough result, which is insensitive to the safety loading coefficient $\delta > 0$.

3 Proofs of Main Results

3.1 Preliminaries

It is well known that, when F is subexponential, the tail of its n-fold convolution is bounded by F's tail in the following way: for any $\varepsilon > 0$, there exists an $A(\varepsilon) > 0$ such that, uniformly for all $n \geq 1$ and all $x \geq 0$,

$$\overline{F^{n*}}(x) \leq A(\varepsilon)(1+\varepsilon)^n \overline{F}(x). \tag{3.1}$$

See Embrechts *et al.* (1997) (p.41-42), Rolski *et al.* (1999) (p.53), Asmussen (2000) (p.255) and references therein for the details of this classical inequality. Note that, in Klüppelberg and Mikosch (1997) Assumption B was assumed in order to use (3.1) in their proof. Tang *et al.* (2001) (Lemma 3.2) found that (3.1) can be improved to

$$\overline{F^{n*}}(x) \leq C n^{\beta+\varepsilon} \overline{F}(x) \tag{3.2}$$

uniformly for all $n \geq 1$ and all $x \geq 0$ if $F \in \mathrm{ERV}(-\alpha, -\beta)$, where C is a positive constant, independent of n and x. Recently, Tang and Yan (2002) further studied the two-sided bounds for the tail of the n-fold convolution of a d.f. F with a dominatedly varying tail. As a special case when $F \in \mathrm{ERV}(-\alpha, -\beta)$ with $1 < \alpha \leq \beta < \infty$, the main result in Tang and Yan (2002) shows a sharper upper bound that

$$\overline{F^{n*}}(x) \leq C(\gamma) n \overline{F}(x) \tag{3.3}$$

holds uniformly for $x > \gamma n$ and $n \geq 1$, where $\gamma > \mu$ is arbitrarily fixed and $C(\gamma)$ is a positive constant, independent of n and x. Inequality (3.3) will be our main tool in establishing Theorem 2.2 under Assumption A.

In the proof of Theorem 2.2, we will need the following result:

Lemma 3.1. *Let $\zeta(t)$ be a stochastic process with a mean function $\mathbb{E}\zeta(t) \to 1$. Then we have*

$$\begin{aligned} & P. \quad \zeta(t) \to_p 1 \\ \Longleftrightarrow \; & Q. \quad \mathbb{E}\zeta(t)\mathbb{I}_{(\zeta(t)-1\geq\theta)} = o(1) \quad \textit{for any } \theta > 0 \\ \Longleftrightarrow \; & R. \quad \mathbb{E}\zeta(t)\mathbb{I}_{(|\zeta(t)-1|\geq\theta)} = o(1) \quad \textit{for any } \theta > 0. \end{aligned}$$

Proof. We will show the following order of implications: $P \Rightarrow R \Rightarrow Q \Rightarrow P$.

1. $P \Rightarrow R$. By the well-known dominated convergence theorem, we obtain, from P, that, for any $\theta > 0$,

$$\lim_{t\to\infty} \mathbb{E}\zeta(t)\mathbb{I}_{(|\zeta(t)-1|<\theta)} = 1.$$

Therefore,

$$\mathbb{E}\zeta(t)\mathbb{I}_{(|\zeta(t)-1|\geq\theta)} = 1 - \mathbb{E}\zeta(t)\mathbb{I}_{(|\zeta(t)-1|<\theta)} = o(1).$$

2. $R \Rightarrow Q$. Trivial.

3. $Q \Rightarrow P$. See Lemma 3.3 of Tang *et al.* (2001). □

By this lemma, we can easily see that not only Assumptions B but also Assumption C_β implies Assumption A.

3.2 Proof of Theorem 2.2

We divide $\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k - (1+\delta)\mu) > x\right)$ into two terms as

$$\left(\sum_{|n-\lambda(t)|\geq\theta\lambda(t)} + \sum_{|n-\lambda(t)|<\theta\lambda(t)}\right)\mathbb{P}\left(S_n - (1+\delta)\mu n > x\right)\mathbb{P}\left(N(t)=n\right) \hat{=} I_1 + I_2, \tag{3.4}$$

where $0 < \theta < 1$ is arbitrarily chosen. First we deal with I_1. By (3.3), we have that

$$\mathbb{P}\left(S_n - (1+\delta)\mu n > x\right) \leq Cn\overline{F}(x + (1+\delta)\mu n) \leq Cn\overline{F}(x).$$

Thus, we obtain

$$\begin{aligned} I_1 &\leq C\overline{F}(x)\mathbb{E}N(t)\mathbb{I}_{(|N(t)-\lambda(t)|\geq\theta\lambda(t))} \\ &= o(\overline{F}(x)\lambda(t)) \\ &= o(\overline{F}(x+\delta\lambda(t)\mu)\lambda(t)), \end{aligned} \tag{3.5}$$

where, in the penult step we have used Lemma 3.1 with $\zeta(t) = N(t)/\lambda(t)$, and in the last step we have used the definition (2.3) and the x-region given in the theorem. Now we deal with I_2. Note that, in this case, we have $x \geq \gamma\lambda(t) \geq (\gamma/(1+\theta))n$. Thus, it follows from Theorem 2.1 that

$$I_2 \sim \sum_{|n-\lambda(t)|<\theta\lambda(t)} n\overline{F}(x+\delta\mu n)\mathbb{P}\left(N(t)=n\right). \tag{3.6}$$

So, by Assumption A and Lemma 3.1 with $\zeta(t) = N(t)/\lambda(t)$, we obtain

$$\begin{aligned}
\frac{I_2}{\overline{F}(x+\delta\lambda(t)\mu)\lambda(t)} &\sim \sum_{|n-\lambda(t)|<\theta\lambda(t)} \frac{\overline{F}(x+\delta\mu n)}{\overline{F}(x+\delta\lambda(t)\mu)} \frac{n}{\lambda(t)} \mathbb{P}\left(N(t)=n\right) \\
&\leq \sup_{x\geq\gamma\lambda(t)} \frac{\overline{F}(x+\delta\mu(1-\theta)\lambda(t))}{\overline{F}(x+\delta\lambda(t)\mu)} \mathbb{E}\frac{N(t)}{\lambda(t)} \mathbb{I}_{\left(\left|\frac{N(t)}{\lambda(t)}-1\right|\leq\theta\right)} \\
&\sim \sup_{x\geq\gamma\lambda(t)} \frac{\overline{F}(x+\delta\mu(1-\theta)\lambda(t))}{\overline{F}(x+\delta\lambda(t)\mu)}. \qquad (3.7)
\end{aligned}$$

By the same approach, we also obtain the corresponding lower bound

$$\inf_{x\geq\gamma\lambda(t)} \frac{\overline{F}(x+\delta\mu(1+\theta)\lambda(t))}{\overline{F}(x+\delta\lambda(t)\mu)}. \qquad (3.8)$$

We will use another equivalent description of ERV: $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $1 < \alpha \leq \beta < \infty$ if and only if

$$y^{-\alpha} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\beta}, \qquad \textit{for any } 0 < y < 1. \qquad (3.9)$$

Using (3.9), and substituting (3.5), (3.6) and (3.7) into (3.4), we obtain that

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k-(1+\delta)\mu)>x\right)}{\overline{F}(x+\delta\lambda(t)\mu)\lambda(t)} \\
&\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{\overline{F}(x+\delta(1-\theta)\lambda(t)\mu)}{\overline{F}(x+\delta\lambda(t)\mu)} \\
&\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{\overline{F}\left((1-\theta)(x+\delta\lambda(t)\mu)\right)}{\overline{F}(x+\delta\lambda(t)\mu)} \\
&\leq (1-\theta)^{-\beta} \to 1, \quad \textit{as } \theta \to 0. \qquad (3.10)
\end{aligned}$$

Symmetrically, by (3.8),

$$\liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k-(1+\delta)\mu)>x\right)}{\overline{F}(x+\delta\lambda(t)\mu)\lambda(t)} \geq (1+\theta)^{-\alpha} \to 1 \quad \textit{as } \theta \to 0. \qquad (3.11)$$

Thus, from (3.10) and (3.11), we finally obtain the desired result (2.9). This ends the proof of Theorem 2.2.

3.3 Proof of Theorem 2.4

For the x-region $x \gg \lambda(t)$, from Corollary 2.3 and Theorem 2.1 in Tang *et al.* (2001), we see that the asymptotics in (2.12) are valid. In the following, we aim to prove the asymptotics in (2.11) for the x-region $x \geq \gamma\lambda(t)$ for any $\gamma > 0$. Clearly, we only need to prove that

$$\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k - \mu) > x\right) \sim \lambda(t)\overline{F}(x) \tag{3.12}$$

holds uniformly for $x \geq \gamma\lambda(t)$ and for any fixed $\gamma > 0$.

As noted at the end of Subsection 3.1, Assumption C_β implies Assumption A. So, it is easy to see that there exists some $\varepsilon(t)$ satisfying $0 < \varepsilon(t) \to 0$ such that

$$\mathbb{P}\left(|N(t)/\lambda(t) - 1| < \varepsilon(t)\right) \to 1 \tag{3.13}$$

(see Klüppelberg and Mikosch (1997)). Now for a given $\varepsilon > 0$, we choose some $l(t)$ such that $0 < l(t) \to \infty$ but $(1+\varepsilon)^{l(t)} = o(\lambda(t))$. We write $\mathbb{P}\left(\sum_{k=1}^{N(t)}(X_k - \mu) > x\right)$ as the sum $\sum_{n=1}^{\infty} \mathbb{P}(S_n - \mu n > x)\mathbb{P}(N(t) = n)$ and divide this sum into five parts:

$$\begin{aligned}\sum_{n=1}^{\infty} &= \sum_{n=1}^{[l(t)]} + \sum_{n=[l(t)]+1}^{[(1-\varepsilon(t))\lambda(t)]} + \sum_{n=[(1-\varepsilon(t))\lambda(t)]+1}^{[(1+\varepsilon(t))\lambda(t)]} + \sum_{n=[(1+\varepsilon(t))\lambda(t)]+1}^{[(1+\theta)\lambda(t)]} + \sum_{n=[(1+\theta)\lambda(t)]+1}^{\infty} \\ &\hat{=}\ J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned} \tag{3.14}$$

Clearly, by the classical inequality (3.1),

$$J_1 \leq A(\varepsilon)\overline{F}(x)\sum_{n=1}^{l(t)}(1+\varepsilon)^n \mathbb{P}\left(N(t) = n\right) \leq A(\varepsilon)\overline{F}(x)\left(1+\varepsilon\right)^{l(t)} = o(\overline{F}(x)\lambda(t)). \tag{3.15}$$

Moreover, noting that the x-region in J_2 and J_4 is that $x \geq \gamma\lambda(t) \geq (\gamma/(1+\theta))n$, we have, from (2.4) and (3.13), that

$$\begin{aligned}J_2 + J_4 &\sim \left(\sum_{n=[l(t)]+1}^{[(1-\varepsilon(t))\lambda(t)]} + \sum_{n=[(1+\varepsilon(t))\lambda(t)]+1}^{[(1+\theta)\lambda(t)]}\right) n\overline{F}(x)\mathbb{P}\left(N(t) = n\right) \\ &\leq (1+\theta)\overline{F}(x)\lambda(t)\mathbb{P}\left(|N(t)/\lambda(t) - 1| \geq \varepsilon(t)\right) \\ &= o(\overline{F}(x)\lambda(t)).\end{aligned} \tag{3.16}$$

As for J_3, by a similar reasoning as above, we have that

$$\begin{aligned}J_3 &\sim \sum_{n=[(1-\varepsilon(t))\lambda(t)]+1}^{[(1+\varepsilon(t))\lambda(t)]} n\overline{F}(x)\mathbb{P}\left(N(t) = n\right) \\ &\sim \lambda(t)\overline{F}(x)\mathbb{P}\left([(1-\varepsilon(t))\lambda(t)] + 1 \leq N(t) \leq [(1+\varepsilon(t))\lambda(t)]\right) \\ &\sim \overline{F}(x)\lambda(t).\end{aligned} \tag{3.17}$$

Finally, we deal with the last term J_5. By Assumption C_β and the inequality (3.2) with $\varepsilon/2$ replacing ε, we deduce

$$J_5 \leq \sum_{n\geq(1+\theta)\lambda(t)} Cn^{\beta+\varepsilon/2}\overline{F}(x)\mathbb{P}\left(N(t)=n\right) = o(\overline{F}(x)\lambda(t)). \tag{3.18}$$

Now, substituting (3.15), (3.16), (3.17) and (3.18) into (3.14), we obtain (3.12). This ends the proof of Theorem 2.4.

3.4 Proof of Theorem 2.5

Clearly, we have the upper bound for $\mathbb{P}\left(\sup_{0\leq t\leq T}\sum_{k=1}^{N(t)}(X_k-(1+\delta)\mu)>x\right)$:

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\sum_{k=1}^{N(t)}X_k>x\right)=\mathbb{P}\left(\sum_{k=1}^{N(T)}X_k>x\right)\sim\lambda(T)\overline{F}(x),$$

here, in the last step, we have used the classical inequality (3.1), Assumption D and the dominated convergence theorem (see for example Theorem A3.20 in Embrechts *et al.* (1997) for the similar discussions). Now we aim to give the corresponding lower bound. For any large $M>0$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq T}\sum_{k=1}^{N(t)}(X_k-(1+\delta)\mu)>x\right)&\geq\mathbb{P}\left(\sum_{k=1}^{N(T)}(X_k-(1+\delta)\mu)>x\right)\\&\geq\mathbb{P}\left(\sum_{k=1}^{N(T)}(X_k-(1+\delta)\mu)>x,\ N(T)\leq M\right)\\&\geq\sum_{n\leq M}\mathbb{P}\left(S_n>x+(1+\delta)\mu M\right)\mathbb{P}\left(N(T)=n\right).\end{aligned}$$

So it follows that, for the given $M>0$,

$$\begin{aligned}&\liminf_{x\to\infty}\frac{\mathbb{P}\left(\sup_{0\leq t\leq T}\sum_{k=1}^{N(t)}(X_k-(1+\delta)\mu)>x\right)}{\lambda(T)\overline{F}(x)}\\&\geq\liminf_{x\to\infty}\sum_{n\leq M}\frac{\mathbb{P}\left(S_n>x+(1+\delta)\mu M\right)}{\overline{F}(x)}\frac{\mathbb{P}\left(N(T)=n\right)}{\lambda(T)}\\&=\liminf_{x\to\infty}\sum_{n\leq M}\frac{\mathbb{P}\left(\sum_{k=1}^{n}X_k>x+(1+\delta)\mu M\right)}{\overline{F}(x+(1+\delta)\mu M)}\cdot\frac{\mathbb{P}\left(N(T)=n\right)}{\lambda(T)}\\&=\frac{\mathbb{E}N(T)\mathbb{I}_{(N(T)\leq M)}}{\lambda(T)},\end{aligned}$$

where we have used the properties (2.1) and (2.2) of the subexponential class. Hence, by letting $M \to \infty$ on the right-hand side of the inequalities above, we also obtain the desired lower bound as $\lambda(T)\overline{F}(x)$. This ends the proof of Theorem 2.5.

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