Simulation

1.1 Inverse Transform Method

The inverse transform method is identical in both the continuous and discrete cases , albeit the presentation is different in the text and there is a difference in their implementation. Essentially, the method is about generating a uniform random variable on the unit interval $(0, 1)$ and using $F^{-1}(U)$ as the simulated random variable from the distribution function F. As mentioned in class, one has to suitably define F^{-1} in the discrete case but this re-definition is **not** important - what is important is to note that the properties in both the discrete and continuous case remain the same.

Some remarks on the method itself:

i. Always applicable, whatever F be - discrete, continuous or even mixed.
ii. Not always efficient - Alternate methods might be faster.
iii. One uniform random variable needed for one simulated random variable from F.
iv. Small values of U will **always** correspond to small values of the simulated variable and vice versa.

Some remarks on variations:

i. Sometimes, like in the case of exponential, the formula for $F^{-1}(U)$ involves finding $1-U$ - this is one *minus operation* which adds to the simulation time, especially significant when one is simulating a billion random variables. And moreover the operation is redundant as $1-U$ has the same distribution as U. **But** the change of $1-U$ to U makes small values of U correspond to large values of the simulated variable. Problems based on this removal of redundancy have appeared quite a few times in the SOA exams.
ii. On occasions you want to generate $Z = \psi(X)$, for some function $\psi(\cdot)$. For example, Z could be the present value random variable corresponding to an insurance product or the loss random variable and X would then be the future life time random variable. There are two ways of using the inverse transform method. First, use it to generate X and use the simulated value of X to get a simulated value of Z by calculating $\psi(X)$. Second, find the distribution of Z and apply the inverse transform method directly to simulate Z. Some comments:

a. If $\psi(\cdot)$ is monotone non-decreasing then both the methods are one and the same. An example, will be the present value random variable from a life annuity product.
b. If $\psi(\cdot)$ is monotone non-increasing then under the first method small values of U correspond to large values of the simulated value of Z. An example, will be the loss random variable from a whole life insurance product.
c. If $\psi(\cdot)$ is **not** monotone then the two methods will be quite distinct. An example, will be the loss random variable from a pure endowment product.

The algorithm for the discrete case: Let X take the values $x_1 < x_2 < \ldots$ with probabilities $p_1, p_2, \ldots$, respectively. Then

$$X = \begin{cases} x_1, & u < p_1; \\ \\ x_j, & \text{if for some } j \geq 2,\ \sum_{k=1}^{j-1} p_k \leq u < \sum_{k=1}^{j} p_k; \end{cases}$$

On a Variation for the Discrete Case The above algorithm can be made more efficient if the $x_i's$ are ordered in decreasing order of $p_i's$, i.e. if x_1 is the mode, x_2 is the value which X assumes with second highest probability and so on...

Acceptance-Rejection Method

This method is about using an auxiliary random variable Y to generate X. The method is not always applicable:

i. **Discrete case:**
   a. Applicable if

$$\Pr(X = x) > 0 \implies \Pr(Y = x) > 0, \forall x \quad \text{and moreover} \quad \max \frac{\Pr(X = x)}{\Pr(Y = x)} < \infty$$

   b. The value c is such that

$$\max \frac{\Pr(X = x)}{\Pr(Y = x)} \leq c \quad \text{and the optimum value satisfies} \quad \max \frac{\Pr(X = x)}{\Pr(Y = x)} = c^*$$

   c. $c \geq 1$ and equal to one if and only if $X \stackrel{\mathrm{d}}{=} Y$. Small values of c are preferred to large values.
   d. The algorithm:
      1. Simulate Y and let the simulated value be y.
      2. Generate U from $U(0, c\Pr(Y = y)) = c\Pr(Y = y)\, U(0, 1)$
      3. If $U < \Pr(X = y)$, then define the simulated value of X as y. Else return to step 1 (i.e. start again).
   e. The probability of rejecting a generated value of Y is $1 - \frac{1}{c}$. The expected value of the number of $Y's$ needed to generate a single value of X is c and the distribution of the number of $Y's$ needed is one plus a geometric with $\beta = c - 1$.

ii. **Continuous Case:**

a. Same as the discrete case except replace $\Pr(X = x)$ by $f(x)$ and $\Pr(Y = y)$ by $g(y)$, where $f(\cdot)$ and $g(\cdot)$ are densities of X and Y respectively.

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