1 Ruin Theory

1.1 Common Notations

The random variable X will always denote the claim random variable; p_1 and p_2 will denote its first and second moments. Also the probability distribution function of X will be denoted by $P(\cdot)$.

u will denote the initial surplus. The probability of ruin when the initial surplus is u will be denote by $\psi(u)$ and will be qualified by a tilde, as $\tilde{\psi}(u)$, when we are referring to the discrete model.

The letter T will denote the time of ruin and will be qualified by a tilde, as $\tilde{T}$, when we are referring to the discrete model. They are defined by

$$T \stackrel{\text{def}}{=} \min\{t : U_t < 0\} \quad \text{and} \quad \tilde{T} \stackrel{\text{def}}{=} \min\{n : U_n < 0\},$$

respectively.

The letter R will denote the adjustment coefficient and will be qualified by a tilde, as $\tilde{R}$, when we are referring to the discrete model.

The letter θ will always denote the relative security loading and is related to c by

$$c = \begin{cases} (1+\theta)\mu, & \text{General Discrete Model;} \\ (1+\theta)\mathbb{E}(N)p_1, & \text{Discrete Model with } W \text{ having a compound distribution;} \\ (1+\theta)\lambda p_1, & \text{Discrete Model with } W \text{ having a compound Poisson distribution;} \\ (1+\theta)\lambda p_1, & \text{Continuous Model;} \end{cases}$$

It is also useful to state the main theorem here to facilitate comparison:

$$\tilde{\psi}(u) = \begin{cases} 1, & \text{if } \mu \geq c \\ \frac{\exp\{-\tilde{R}u\}}{\mathbb{E}\left(\exp\{-\tilde{R}U_{\tilde{T}}\}|\tilde{T}<\infty\right)} & \text{if } \mu < c; \end{cases} \quad \text{and} \quad \psi(u) = \begin{cases} 1, & \text{if } \mu \geq c \\ \frac{\exp\{-Ru\}}{\mathbb{E}\left(\exp\{-RU_T\}|T<\infty\right)} & \text{if } \mu < c; \end{cases}$$

Note that the probability of ruin will be always positive whatever the value of the initial surplus, even though it will be decreasing with u.

1.2 Discrete Model

The model is described by

$$U_n = u + nc - \sum_{i=1}^{n} W_i, \quad \text{for } n \geq 0,$$

where c is the premium **per period** and W_i's are i.i.d. W, where W represents the claims **per period**. The adjustment coefficient, $\tilde{R}$, defined as the smallest **positive** root of the equation

$$\ln M_W(r) = rc.$$

The expectation and variance of W are denoted by μ and σ^2. If W is normal, then

$$\tilde{R} = \frac{2(c-\mu)}{\sigma^2},$$

a formula which becomes an approximation in general. The approximation simplifies to,

$$\tilde{R} \approx \frac{2\theta p_1 \mathbb{E}(N)}{(p_2 - p_1^2)\mathbb{E}(N) + p_1^2 \mathrm{Var}\,(N)}$$

if W has a compound distribution.

Types of Problems

i. Note that in the past four years the discrete case has not appeared explicitly as there is no discussion, in the text, of the cases where the ruin probability can be written in a closed form. But remember that an upper bound for the ruin probability is given by $\exp\{-\tilde{R}u\}$.
ii. The only types of problems on the discrete case can be something conceptual, as in the problem on page 63 or problems involving the adjustment coefficient, like the problem on page 60, or something using ad hoc methods of analysis like the problem on page 56.

1.3 Continuous Model

The model is described by

$$U_t = u + ct - S_t, \quad \text{for } n \geq 0,$$

where c is the continuous premium rate and S_t is a compound Poisson process. In this model there are two random variables which are important. First, the amount by which the surplus drops below the **initial level** given that the surplus does drop below the initial level, denoted by L_1. Second, the maximum aggregate loss, denoted by L, which satisfies,

$$L = \max_{t \geq 0}\{S_t - ct\} = u - \min_{t \geq 0} U_t$$

The interesting facts about L_1 are

$$f_{L_1}(x) = \begin{cases} \frac{1-P(x)}{p_1}, & x \geq 0; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad M_{L_1}(r) = \frac{M_X(r) - 1}{p_1 r}$$

and

$$\mathbb{E}(L_1) = \frac{p_2}{2p_1} \quad \text{and in general } \mathbb{E}\left(L_1^k\right) = \frac{p_{k+1}}{(k+1)p_1},\ k \geq 1.$$

The foremost interesting fact about L is that it has a compound distribution with the frequency being a Geometric random variable with the parameter β (as in the SOA tables) being θ^{-1} and the severity being distributed as L_1. This representation gives us the moments of L in terms of the moments of L_1 and hence in terms of moments of X. Moreover, L is a mixed type random variable with

$$\Pr(L = 0) = \frac{\theta}{(1+\theta)} = 1 - \psi(0).$$

Also, interesting is the more general fact that

$$\Pr(L \leq u) = 1 - \psi(u).$$

An interesting form of the m.g.f. of L deriving from it having a compound distribution is

$$M_L(r) = (1 - \psi(0)) + \psi(0)\frac{\theta M_{L_1}(r)}{(1+\theta) - M_{L_1}(r)}$$

The adjustment coefficient, R, defined as the smallest **positive** root of the equation

$$M_X(r) = (1+\theta)p_1 r + 1.$$

In this model we have

$$\psi(0) = \frac{1}{(1+\theta)},$$

even though for other values of u we do not have a universal closed form solution. But there are cases where we have closed form solutions, which are listed below.

The case of Exponential

There are two useful ways to remember the formula. The first one is to remember that the adjustment coefficient is given by

$$R = \frac{\theta}{(1+\theta)p_1}$$

and the ruin probability by

$$\psi(u) = \left(\frac{1}{1+\theta}\right)\exp\{-Ru\}$$

The second is the form

$$\psi(u) = \left(\frac{1}{1+\theta}\right)\exp\left\{-\frac{2\theta p_1 u}{(1+\theta)p_2}\right\},$$

which simplifies to the above form as for the exponential $p_2 = 2p_1^2$. The advantage of the latter form is that it is an approximation of the ruin probability in the **general** case.

Both the forms involve p_1 as it makes the formulae invariant with respect to the parametrization that one uses for the exponential.

The case of X being a Mixture of Exponentials

In this case, simplify

$$M_L(r) - (1-\psi(0)) = \left(\frac{1}{1+\theta}\right)\frac{\theta[M_X(r)-1]}{(1+\theta)p_1 r - [M_X(r)-1]} \quad \text{to the form} \quad \sum_{i=1}^{n} C_i \frac{r_i}{r_i - r}$$

where n will represent the number of exponential components in the mixture. From the above simplified representation, we have

$$\psi(u) = \sum_{i=1}^{n} C_i \exp\{-r_i u\} \quad \sum_{i=1}^{n} C_i = \frac{1}{1+\theta} \quad \text{and } R = \min_{1\le i\le n} r_i$$

Types of Problems

i. Problems relating to the probability of ruin could be of three possible types - with $u = 0$ like the problem on page 58, or with X being an exponential or in a rare chance (my subjective feeling) involving an X which is a mixture of exponentials. Also, if a closed form does not exist you might be asked to use the above approximation.

ii. One involving the distribution of L_1, like the problems on page 57 and 65. Note that in the case that X is a constant x, you get L_1 being an uniform $U(0, x)$.

iii. Problems involving L - Note that the distribution of L is completely specified if and only if so is the ruin probability. Hence, in the case of X being an exponential, L is a mixed distribution with mass at 0 and the continuous part being an exponential with mean $(1+\theta^{-1})p_1$. In the case of X being a mixture of exponentials, so is the continuous part a mixture of exponentials.

iv. Always, there can be ad hoc problems like those on page 56 and 66.

v. Note that probability of the surplus going below the initial level is same as $\psi(0)$.