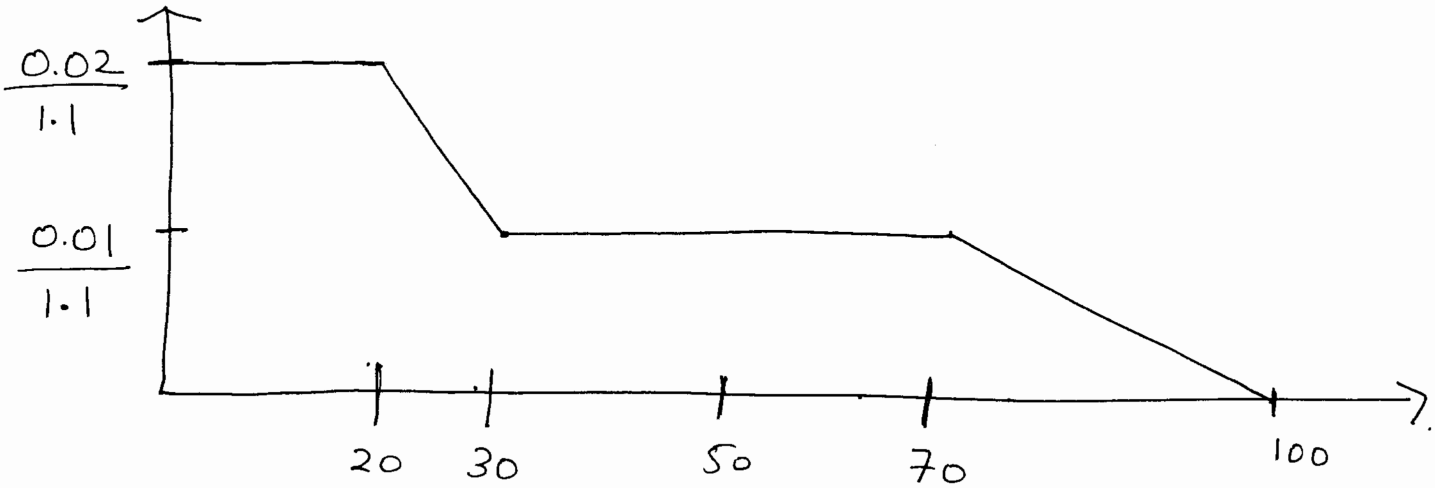
Example Find the expectation of the random variable whose density is graphed below.



Solution 1 [Brute Force]

From the graph, it is easy to write down the density of, say X, as

$$f(x) = \begin{cases} \dfrac{0.02}{1.1} & 0 \le x < 20 \\[2ex] \dfrac{0.02}{1.1} - \dfrac{(x-20)}{10}\,\dfrac{0.01}{1.1} & 20 \le x < 30 \\[2ex] 0.01/1.1 & 30 \le x < 70 \\[2ex] \dfrac{0.01}{1.1} - \dfrac{(x-70)}{30}\,\dfrac{0.01}{1.1} & 70 < x \le 100 \\[2ex] 0 & \text{otherwise.} \end{cases}$$

The above is easy as $f(\cdot)$ is piecewise linear.

Now for EX, we could use the formula $\int x f(x)\,dx$ as shown below.

$$EX = \int x\, f(x)\, dx$$

$$= \int_0^{20} \frac{0.02}{1.1}\, x\, dx \qquad \left[\text{Can be shown to be equal to } \frac{4}{1.1}\right]$$

$$+ \int_{20}^{30} \left[\frac{0.02}{1.1} - \frac{(x-20)}{10}\,\frac{0.01}{1.1}\right] x\, dx \qquad \left[\frac{11}{3*1.1}\right]$$

$$+ \int_{30}^{70} \frac{0.01}{1.1}\, x\, dx \qquad \left[\frac{20}{1.1}\right]$$

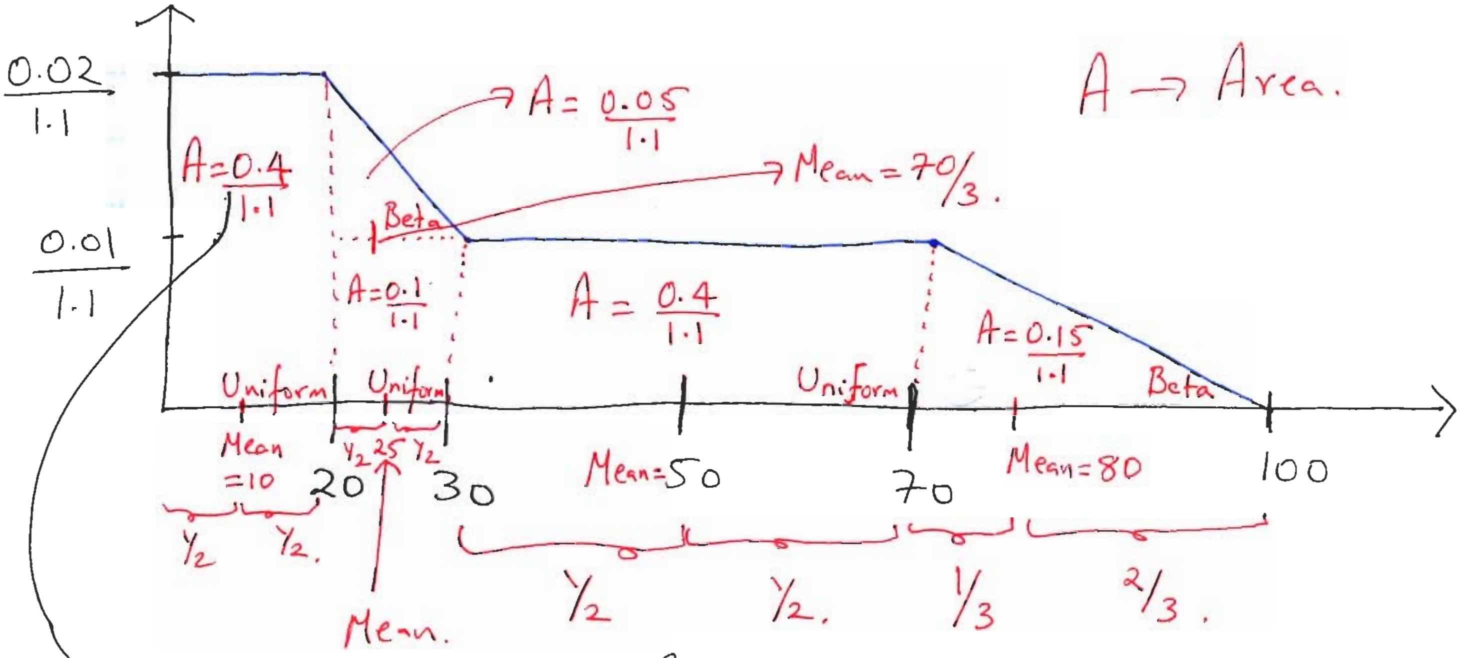
$$+ \int_{70}^{100} \left[\frac{0.01}{1.1} - \frac{(x-70)}{30}\,\frac{0.01}{1.1}\right] x\, dx \qquad \left[\frac{12}{1.1}\right]$$

$$= \frac{4}{1.1} + \frac{11}{3*1.1} + \frac{20}{1.1} + \frac{12}{1.1} = \underline{\underline{36.06}}$$

Solution 2

Mixture of shifted and scaled Be(1,2) and U(0,1).

We shall use the fact that $E(Be(1,2)) = \frac{1}{3}$.



[These are areas]

$$X \begin{cases} \xrightarrow{\frac{0.4}{1.1}} U(0,20) \overset{d}{=} 20\,U(0,1) \\ \xrightarrow{0.1/1.1} U(20,30) \overset{d}{=} 20 + 10\,U(0,1) \\ \xrightarrow{0.05/1.1} 20 + 10\,Be(1,2) \\ \xrightarrow{0.4/1.1} U(30,70) \overset{d}{=} 30 + 40\,U(0,1) \\ \xrightarrow{\frac{0.15}{1.1}} 70 + 30\,Be(1,2) \end{cases}$$

scales

shifts

Hence, $EX = \frac{0.4}{1.1} * 10 + \frac{0.1}{1.1} * 25 + \frac{0.05}{1.1} * \frac{70}{3}$

$$+ \frac{0.4}{1.1} * 50 + \frac{0.15}{1.1} * 80 = \underline{\underline{36.\overline{06}}}$$

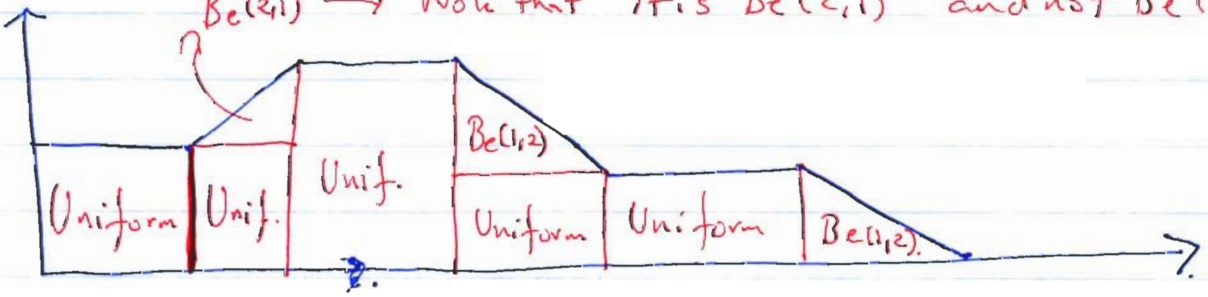
The above can be "read off" the graph or using the representation above.

The advantage of this method is phenomenal in terms of insight and also when it comes to computation of higher moments, for e.g.

<u>Variance</u>. [Qn. The above has five components - Can you slice and dice differently to come up with only four?]

In Summary, piecewise linear densities lead to mixtures ~~shapes like~~ of ~~Be(1,2), Unif~~

$Be(2,1)$ → Note that it is $Be(2,1)$ and not $Be(1,2)$.



shifted and scaled $Be(1,2)$, $Be(2,1)$ and $U(0,1)$.

The study note would refer to the above as "splicing".