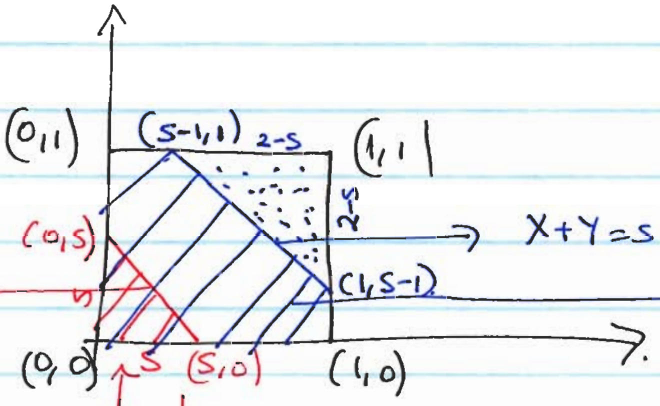
Finding the distribution of

$X+Y$ when X and Y are iid $U(0,1)$.

Method 1: Geometric.

Figure 1



the line represent $X+Y=s$ where $0<s<1$

$X+Y=s$ where $1 \le s < 2$.

Area will be equal to area of the unit square minus the area of the dotted triangle.
$= 1 - \frac{1}{2}(2-s)^2$

area $= \frac{1}{2} s^2$ where $0<s<1$

Hence the distbn. fn. of $X+Y$ is given by

$$F_{X+Y}(s) = \Pr(X+Y \le s) = \begin{cases} 0 & s<0 \\ \frac{s^2}{2} & 0<s<1 \\ 1-\frac{1}{2}(2-s)^2 & 1 \le s < 2 \\ 1 & s \geqslant 2. \end{cases}$$

Check: The resulting fn. should be continuous, and it is as can be seen to be at points $s = 1$ and 2.

The density then, is the derivative of F_{X+Y} which is given by

$$f_{X+Y}(s) = \begin{cases} s & 0 < s < 1 \\ 2-s & 1 \leq s < 2 \\ 0 & \text{otherwise.} \end{cases}$$

i.e.

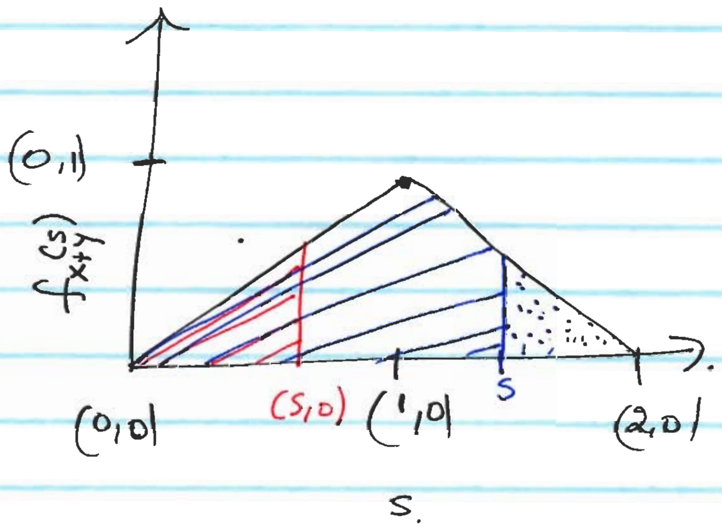


Figure 2.

One can directly argue from figure ① to figure ②.

Can you?

Method 2 Brute Force

$$f_{X+Y}(s) = \int_0^1 f_X(t)\, f_Y(s-t)\, dt \quad , \quad \forall\, s \in [0,2]$$

The limits are 0 to 1 as $f_X(t)$ is positive only between these limits.

But that would be the same with f_Y too. Hence

$$0 < s-t < 1$$

$$\Leftrightarrow \quad s-1 < t < s.$$

Hence combining the limits on t, we have

$$(s-1) \vee 0 < t < s \wedge 1 \qquad \vee \to \text{maximum}$$
$$\wedge \to \text{minimum}.$$

Hence $$f_{X+Y}(s) = \int_{(s-1)\vee 0}^{s \wedge 1} f_X(t)\, f_Y(s-t)\, dt.$$

But as the densities are ~~one~~ equal to 1 when positive we have

$$f_{X+Y}(s) = \int_{(s-1)\vee 0}^{s\wedge 1} dt = \left[s\wedge 1 - (s-1)\vee 0 \right]$$

$$= \begin{cases} 0 & s<0 \quad \text{or} \quad s \geq 2 \\ s-0 = s & 0 \leq s < 1 \\ 1-(s-1) = 2-s. & 1 \leq s < 2 \end{cases}$$