Homework #4 Solution

12.15 S : compound Poisson dist^n w/ λ & $p(x)$ (discrete), $x>0$

Let $0<\alpha<1$.

$\tilde{S}$: Compound Poisson w/ $\tilde{\lambda} = \dfrac{\lambda}{\alpha}$

$$\tilde{p}(x) = \begin{cases} \alpha\, p(x) & x>0 \\ 1-\alpha & x=0 \end{cases}$$

a. Let $\tilde{X}$ be a random variable associated with $\tilde{p}(x)$.

$$M_{\tilde{X}}(t) = E\left[e^{t\tilde{X}}\right]$$

$$= e^{t\cdot 0}\,\tilde{p}(0) + \sum_{x>0} e^{tx}\cdot \tilde{p}(x)$$

$$= (1-\alpha) + \alpha M_X(t) = \alpha\left(M_X(t)-1\right)+1$$

$$\therefore\ M_{\tilde{S}}(t) = e^{\tilde{\lambda}\left(M_{\tilde{X}}(t)-1\right)}$$

$$= e^{\frac{\lambda}{\alpha}\cdot \alpha\left(M_X(t)-1\right)} = M_S(t)$$

$\therefore$ S & $\tilde{S}$ have the same distribution.

b. Suppose $x_1, \cdots, x_r$ are st $\sum p(x_i)=1$

and N_i be the number of claims equal to x_i

Then $S = x_1 N_1 + \cdots + x_r N_r$

$N_i \sim \text{Poisson}(\lambda_i)$, $\lambda_i = \lambda p(x_i)$.

For $\tilde{S}$ we have

$$\tilde{S} = 0 N_0 + x_1 N_1 + \cdots + x_r N_r$$

$$N_i \sim \text{Poisson}(\tilde{\lambda}_i)$$

$$\tilde{\lambda}_i = \tilde{\lambda} \Pr(\tilde{X} = x_i)$$

$$= \frac{\lambda}{\alpha}(\alpha p(x_i)) = \lambda p(x_i) = \lambda_i$$

for $x_i > 0$.

$\therefore$ S & $\tilde{S}$ have the same distribution

12.17 Use Thm 12.4.3

$$f_S(x) = \sum_{i=1}^{x} \left[a + \left(\frac{bi}{x} \right) \right] p(i) f_S(x-i) \qquad x = 1, 2, \ldots$$

$$f_S(0) = \Pr(N = 0)$$

$p(1) = 0.7$ $p(2) = 0.3$

a. Use (12.4.16)
b. Use (12.4.17)
c. Use (12.4.18)

See below for values of $f_S(\cdot)$

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	a	b	c
$f_S(0)$	0.011109	0.0442	0.00195
$f_S(1)$	0.034993	0.0696	0.01230
$f_S(2)$	0.070111	0.0968	0.03928
$f_S(3)$	0.1051	0.1082	0.08580
$f_S(4)$	0.1301	0.1110	0.13777
$f_S(5)$	0.138923	0.1050	0.17366

d. (a) mean $= \lambda = 4.5$ Variance $= \lambda = 4.5$

(b) mean $= \frac{rq}{p} = 4.5$ Variance $= \frac{rq}{p^2} = 9$

(c) mean $= mp = 4.5$ Variance $= mp(1-p) = 2.25$

2.18 S : Compound Negative Binomial Distribution w/ r, p.

$\pi_i = p(x_i)$

$S = x_1 N_1 + \cdots + x_m N_m$.

a. Joint Moment Generating Function of $N_1, \cdots, N_m$

$$= M_{N_1,\cdots,N_m}(t_1,\cdots,t_m)$$

$$= E\left(e^{t_1N_1+\cdots+t_mN_m}\right) = \sum_{n=0}^{\infty} E\left(e^{t_1N_1+\cdots+t_mN_m} \middle| N=n\right) Pr(N=n)$$

$$= \sum_{n=0}^{\infty} \left(\pi_1 e^{t_1} + \cdots + \pi_m e^{t_m}\right)^n \binom{r+n-1}{n} p^r q^n$$

MGF of N_i can be obtained by $M_{N_1,\cdots,N_m}(0,\cdots t_i, 0 \cdots 0)$ ↑ i th

$$M_{N_i}(t_i) = \sum_{n=0}^{\infty} \left(\pi_i e^{t_i} + (1-\pi_i)\right)^n \binom{r+n-1}{n} p^r q^n$$

$$= \sum_{n=0}^{\infty} \binom{r+n-1}{n} p^r \left(q\left(1-\pi_i+\pi_i e^{t_i}\right)\right)^n$$

$$= \left(\frac{p}{1-q+q\pi_i-q\pi_i e^{t_i}}\right)^r$$

$$= \left(\frac{p}{p+q\pi_i-q\pi_i e^{t_i}}\right)^r = \left(\frac{\frac{p}{p+q\pi_i}}{1-\frac{q\pi_i}{p+q\pi_i}e^{t}}\right)^r$$

This is the MGF for Neg. Binomial w/ parameter r & $\frac{p}{p+q\pi_i}$

b. N_1 & N_2 : independent $\Leftrightarrow M_{N_1,N_2}(t_1,t_2) = M_{N_1}(t_1)M_{N_2}(t_2)$

However,

$$M_{N_1N_2}(t_1\ t_2) = M_{N_1,N_2,\cdots,N_m}(t_1,t_2,0,\cdots,0)$$

$$\neq M_{N_1}(t_1)\cdot M_{N_2}(t_2)$$

Hence they are not independent.

12.2) $S_1 \sim$ Compound Poisson w/ λ & $P_1(x)$

Let $S_1 = X_1 + X_2 + \cdots + X_N$ Poisson where $X \sim P_1$

$S_2 \sim$ Compound Neg. Binomial w/ r, p & $P_2(x)$

Let $S_2 = Y_1 + \cdots + Y_M$ Neg. Bin $Y \sim P_2$

$$p_1(x) = \frac{\sum_{k=1}^{\infty} \left(q^k / k \right) \cdot p_2^{*k}(x)}{-\log p}$$

From this, we see that X is distributed as compound "Logarithm", where "Logarithm" distribution has a p.m.f of $\frac{q^k/k}{-\log p}$.

i.e. $X = Y_1 + \cdots + Y_L$ $Y \sim P_2$

"Logarithm"

$\therefore$ $M_X(t) = MGF_L \left(\log M_Y(t) \right)$

$$M_L(t) = \sum_{x=0}^{\infty} \frac{e^{tx} \cdot q^x / x}{-\log p} = \frac{-\log_e(1 - q e^t)}{-\log p}$$

$$= \frac{+\log(1 - q M_Y(t))}{+\log p}$$

Hence $M_{S_1}(t) = e^{\lambda(M_X(t) - 1)}$

$$= e^{(-r\log p)\left(\frac{\log(1 - q M_Y(t))}{\log p} - 1 \right)}$$

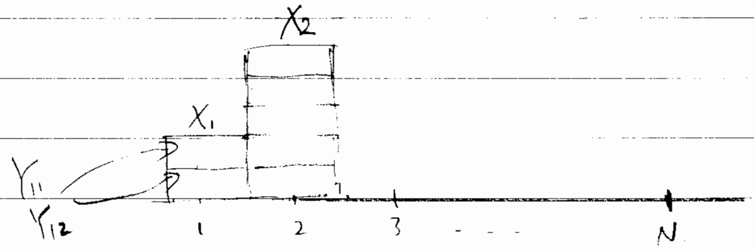
if $\lambda = -r\log p$

$$= e^{(-r)\left(\log \frac{1 - q M_Y(t)}{p} \right)}$$

$$= \left(\frac{p}{1 - q M_Y(t)} \right)^r$$

$$= M_{S_2}(t).$$

(Or) We can see the result as follows.



$$X_i = Y_{i1} + \cdots + Y_{iL_i}$$ from the relation $p_1(x)$ & $p_2(x)$

$$S_1 = Y_{11} + Y_{12} + \cdots + Y_{1L_1} + Y_{21} + \cdots + Y_{2L_2} + \cdots + Y_{N1} + \cdots + Y_{NL_N}$$

Total number of claims

$$= L_1 + L_2 + \cdots + L_N$$

L follows Logarithm & N follows Poisson

$\Longrightarrow$ Negative Binomial

By prob 12.8 w/ $\lambda = -r \log p$

$\therefore$ S_1 is compound Neg. Binomial

w/ $p_2(x)$ as claim distⁿ

i.e. S_1 & S_2 have the same distribution