Homework #3. Solution.

1. $X \sim$ exponential with mean λ.

$d > 0$ is given

$$X \wedge d \begin{cases} X & \text{if } X \le d \\ d & \text{if } X > d \end{cases}$$

① $$E(X \wedge d) = \int_0^d x f_X(x)\,dx + \int_d^\infty d f_X(x)\,dx$$

$$= \int_0^d x \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx + d \Pr(X > d)$$

$$\int_0^d x \cdot \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx = -x e^{-\frac{x}{\lambda}}\Big|_0^d + \int_0^d e^{-\frac{x}{\lambda}}\,dx$$

$$= \cancel{-d e^{-\frac{d}{\lambda}}} + \lambda(1 - e^{-\frac{d}{\lambda}}) + \cancel{d e^{-\frac{d}{\lambda}}} = \lambda(1 - e^{-\frac{d}{\lambda}})$$

② $$E((X \wedge d)^2) = \int_0^d x^2 \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx + \int_d^\infty d^2 f_X(x)\,dx$$

$$= x^2(-e^{-\frac{x}{\lambda}})\Big|_0^d + 2\int_0^d x e^{-\frac{x}{\lambda}}\,dx + d^2 e^{-\frac{d}{\lambda}}$$

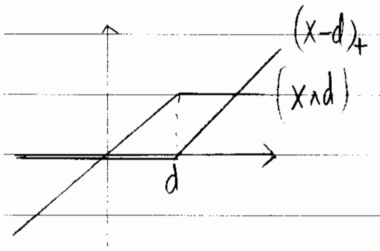
$$= \cancel{-d^2 e^{-\frac{d}{\lambda}}} + 2\lambda\underbrace{\left(-d e^{-\frac{d}{\lambda}} + \lambda(1 - e^{-\frac{d}{\lambda}})\right)}_{\text{part ①}} + \cancel{d^2 e^{-\frac{d}{\lambda}}}$$

$$= 2\lambda^2 - 2\lambda^2 e^{-\frac{d}{\lambda}} - 2\lambda d e^{-\frac{d}{\lambda}}$$

$$Var(X \wedge d) = 2\lambda^2 - 2\lambda^2 e^{-\frac{d}{\lambda}} - 2\lambda d e^{-\frac{d}{\lambda}} - \lambda^2(1 - e^{-\frac{d}{\lambda}})^2$$

$$= \lambda^2 - 2\lambda d e^{-\frac{d}{\lambda}} - \lambda^2 e^{-\frac{2d}{\lambda}}$$

③ Use $(X\wedge d) + (X-d)_+ = X$.



By taking expectation on both sides,

$$E(X\wedge d) + E(X-d)_+ = \lambda.$$

$$\therefore E(X-d)_+ = \lambda - E(X\wedge d) = \lambda - \lambda\left(1-e^{-\frac{d}{\lambda}}\right)$$

$$= \lambda e^{-\frac{d}{\lambda}}$$

④ Since $(X\wedge d) + (X-d)I(X>d) = X$,

by squaring,

$$X^2 = (X\wedge d)^2 + \underbrace{(X-d)^2 I(X>d)}_{\{(X-d)_+\}^2} + \underbrace{2(X\wedge d)(X-d)I(X>d)}_{2d(X-d)_+}$$

Taking expectation,

$$2\lambda^2 = EX^2 = \underbrace{E(X\wedge d)^2}_{2\lambda^2-2\lambda^2e^{-\frac{d}{\lambda}}-2\lambda d e^{-\frac{d}{\lambda}}} + E\{(X-d)_+\}^2 + \underbrace{2d\,E(X-d)_+}_{2d\lambda e^{-\frac{d}{\lambda}}}$$

$$\therefore E\{(X-d)_+\}^2 = 2\lambda^2 e^{-\frac{d}{\lambda}}$$

$$\therefore Var\left((X-d)_+\right) = 2\lambda^2 e^{-\frac{d}{\lambda}} - \lambda^2 e^{-\frac{2d}{\lambda}}$$

$$= \lambda^2 e^{-\frac{d}{\lambda}}\left(2 - e^{-\frac{d}{\lambda}}\right)$$

2. $E[I_d(x)] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_d^\infty (x-d)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$

$z = \frac{x-\mu}{\sigma}$

$\beta = \frac{d-\mu}{\sigma}$

$x-d = \sigma(z-\beta)$

$dx = \sigma dz$

$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_\beta^\infty \sigma(z-\beta)\exp\left(-\frac{z^2}{2}\right)\sigma\, dz$

$= \sigma\left[\frac{1}{\sqrt{2\pi}}\int_\beta^\infty z\exp\left(-\frac{z^2}{2}\right)dz - \underbrace{\frac{\beta}{\sqrt{2\pi}}\int_\beta^\infty \exp\left(-\frac{z^2}{2}\right)dz}_{\beta(1-\Phi(\beta))}\right]$

Since $\int_\beta^\infty z\exp\left(-\frac{z^2}{2}\right)dz = \int_\beta^\infty d\left(-\exp\left(-\frac{z^2}{2}\right)\right) = \exp\left(-\frac{\beta^2}{2}\right)$

$E[I_d(x)] = \sigma\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\beta^2}{2}\right) - \beta(1-\Phi(\beta))\right].$

12.8. S : Compound Poisson w/ Poisson parameter λ

$p(x) = \underbrace{\left[-\log(1-c)\right]^{-1}}_{\text{positive.}}\frac{c^x}{x}, \quad x=1,2,\cdots \quad 0<c<1$

& check $\sum_{x=1}^{\infty} p(x) = 1$.

From (12.3.5)

$M_S(t) = e^{\lambda(M_X(t)-1)}$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}\left[(-\log(1-c))^{-1}\frac{c^x}{x}\right]$$

$$= (-\log(1-c))^{-1}\sum_{x=1}^{\infty}\frac{(ce^t)^x}{x}$$

for small t: $ce^t < 1$ $\rightarrow$ $= (-\log(1-c))^{-1}(-\log(1-ce^t))$

$$M_X(t) - 1 = (-\log(1-c))^{-1}\left[-\log(1-ce^t) + \log(1-c)\right]$$

$$= \log\left(\frac{1-c}{1-ce^t}\right)^{(-\log(1-c))^{-1}}$$

$$\therefore\ M_S(t) = e^{\lambda \log\left(\frac{1-c}{1-ce^t}\right)^{(-\log(1-c))^{-1}}}$$

$$= \left(\frac{1-c}{1-ce^t}\right)^{\lambda(-\log(1-c))^{-1}}$$

mgf of Negative Binomial $= \left(\dfrac{p}{1-qe^t}\right)^r$

By comparing these two,

S follows N.B w/ $p = 1-c$

$r = \lambda(-\log(1-c))^{-1}$

12.10 Let S : number of accidents that result in a claim payment

Then $S = \sum_{i=1}^{N} X_i$ ~~where $N \sim$ Poisson (λ)~~

$$X = \begin{cases} 1 & \text{if damage > deductible} \\ 0 & \text{o.w.} \end{cases}$$

i.e. $p(x_i) = \begin{cases} p & \text{if } x_i = 1 \\ 1-p & \text{if } x_i = 0 \end{cases}$

$$M_X(t) = E(e^{tX}) = e^{t} \cdot p + e^{0} \cdot (1-p) = p(e^{t}-1) + 1$$

$$M_S(t) = e^{\cancel{\lambda(M_X(t)-1)}}$$
$$= e^{\lambda p(e^{t}-1)}$$

i.e $M_S(t)$ is a m.g.f of Poisson distribution with parameter λp.