COURSE NO.: **22S:175** H.W. NO.: **I**
COURSE TITLE: **Risk Theory** H.W. DATE: **8/25/04**
INSTRUCTOR: **Shyamal** DUE DATE: **9/1/04**

Instructions

i. This assignment contains some problems in elementary probability to give you an idea of the probabilistic pre-requisites for this course.
ii. Parts carrying extra credits and starred problems are not required part of the assignment. Nevertheless, a serious attempt followed by a discussion of these during office hours is encouraged.
iii. Constructive comments on the assignments and for that matter any other aspect of the course will be welcomed.

Problem 1 In the following $B(1, 0.5)$ is a symmetric Bernoulli random variable.

1. Let $X = 2(\mathrm{B}(1, 0.5) - 0.5)$ and $Y = X^2$.
   a. Show that $\mathrm{Cov}\,(X, Y) = 0$.
   b. Show that the two random variables are independent.
2. Let X be such that

$$\mathrm{P}\{X = x\} = \begin{cases} 0.25 & x = -2, -1, 1 \text{ and } 2; \\ 0 & \text{otherwise.} \end{cases}$$

   and $Y = X^2$.
   a. Show that $\mathrm{Cov}\,(X, Y) = 0$.
   b. Show that the two random variables are dependent.

Problem 2 This problem explains why for random number generation, computers need to only be able to generate uniform random variables. Let F be a distribution function.

a. Define $\mathrm{F}^{-1}(u) = \inf\{z|\mathrm{F}(z) \geqslant u\}$, where inf stands for infimum. If you have not been exposed to infimum, then assume that for the problem it is the same as the minimum. And if you have been exposed, prove that the infimum is attained here for extra credit. Prove that

$$\{u|\mathrm{F}^{-1}(u) \leqslant x\} = (-\infty, \mathrm{F}(x)],\ \forall u \in (0, 1)$$

   Hint:

$$\begin{aligned} u \leqslant \mathrm{F}(x) &\Rightarrow x \in \{z|\mathrm{F}(z) \geqslant u\} \\ &\Rightarrow x \geqslant \inf\{z|\mathrm{F}(z) \geqslant u\} \\ &\Longleftrightarrow x \geqslant \mathrm{F}^{-1}(u),\ \forall u \in (0, 1) \end{aligned}$$

b. Let U be a $\mathrm{U}(0, 1)$ distributed random variable. Show that $\mathrm{F}^{-1}(U)$ has F as it's distribution function.
c. Why did we have to define the *inverse* of F?

Problem 3 In the following assume the existence of all required moments.

i. It is easy to see that

$$\mathrm{Var}\,(X) = \min_{a \in \mathbb{R}} \mathbb{E}(X - a)^2$$

   a. Prove using Calculus
   b. Prove without using Calculus
ii. Let X and Y be two random variables.

a. Show that $X - \mathbb{E}(X|Y)$ is uncorrelated with $\mathbb{E}(X|Y)$
b. Hence or otherwise show that

$$\begin{aligned} \mathrm{Var}\,(X) &= \mathbb{E}(X - \mathbb{E}(X|Y))^2 + \mathbb{E}(\mathbb{E}(X|Y) - \mathbb{E}(X))^2 \\ \mathrm{Var}\,(X) &= \mathbb{E}(\mathrm{Var}\,(X|Y)) + \mathrm{Var}\,(\mathbb{E}(X|Y)) \end{aligned}$$

c. Can you explain it as a **Pythagoras Theorem** - This part is for extra credit.

Problem 4 For the following functions,

i. Check if they satisfy the required properties of a distribution function.
ii. If one is found to be a distribution function then classify it as either discrete, continuous or a mixture of a discrete distribution and a continuous distribution.
iii. In the case that of a mixture of a discrete distribution and a continuous distribution, decompose it into its constituent discrete and continuous components.
iv. Find the expectation and variance of the random variable associated with the distribution function.

A.

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1)}{24} & i \le x \le i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

B.

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1; \\ \frac{i(i+1)}{12} & i \le x < i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

C.

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1)}{12} & i \le x < i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

D.

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1) + 6}{24} & i \le x < i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

Problem* 1 Let X be a random variable taking values in $\{1, 2, \ldots\}$ such that

$$\mathrm{P}\{X = n\} \propto \begin{cases} \frac{1}{n^2 \log(n)(\log(\log(n)))^2} & n = 5, 6, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

a. Show that $\mathbb{E}(X) < \infty$
b. Show that $\mathbb{E}\left(X^{1+\epsilon}\right) = \infty, \quad \forall \epsilon > 0$