Homework #1 Solution

Problem 1.

$$B(1, 0.5) \begin{cases} 1 & \text{w.p} \quad 1/2 \\ 0 & \text{w.p} \quad 1/2 \end{cases}$$

(5pts)

1. $$X = 2(B(1,0.5) - 0.5) \begin{cases} 1 & \text{w.p} \quad 1/2 \\ -1 & \text{w.p} \quad 1/2 \end{cases}$$

$$Y = X^2 = 1 \quad \text{w.p} \quad 1$$

a. $$\mathrm{Cov}(X,Y) = \underbrace{EX^3}_{0} - \underbrace{EX}_{0}\,\underbrace{EX^2}_{1} = 0$$

b. $$\Pr(X=i, Y=1) = \Pr(X=i)$$
$$= \Pr(X=i)\Pr(Y=1) \qquad \forall\, i = 1, -$$

$\therefore$ X & Y are independent

(5pts)

2. $$X \begin{cases} 2 & \text{w.p.} \quad 1/4 \\ 1 & \text{"} \\ -1 & \text{"} \\ -2 & \text{"} \end{cases} \qquad Y = X^2 \begin{cases} 4 \\ 1 \end{cases}$$

a. It can be easily seen that $EX = 0$ and $EX^3 = 0$

$\therefore$ $\mathrm{Cov}(X,Y) = EX^3 - EX\, EX^2 = 0$

b. $$\Pr(X=1, Y=4) = \Pr(\underbrace{X=1,\ X^2=4}_{\text{This cannot happen.}}) = 0$$

However, $\Pr(X=1)\Pr(Y=4) = \left(1/4\right) \cdot \left(1/2\right) = 1/8$

$\therefore$ X & Y are dependent

Problem 2. a. ① Infimum is attained $\Leftrightarrow$ For any given $u \in (0,1)$
$\{z \mid F(z) \geq u\} = [a, \infty)$ for some a
closed

(4 pts)

(+ extra 2 pts)

This results from the right continuity of F.

Suppose $\{z \mid F(z) \geq u\} \supseteq (a, \infty)$

Then $F(a+\varepsilon) \geq u \quad \forall \varepsilon > 0$

& $F(a+) \geq u$

$\quad\; \| $

$F(a) \qquad \therefore\ a \in \{z \mid F(z) \geq u\}$.

② From Hint, it follows that $\{u \mid F^{-1}(u) \leq x\} \supseteq (-\infty, F(x))$

③ Let $v \in \{u \mid F^{-1}(u) \leq x\}$

$F^{-1}(v) \leq x$

$v \leq F(F^{-1}(v)) \leq F(x)$

By definition of F^{-1} & the fact that infimum is attained.

$\therefore \{u \mid F^{-1}(u) \leq x\} \subseteq (-\infty, F(x))$

By ②, ③, the two sets are equal.

(3 pts) b. $U \sim U(0.1)$

Distribution ftn of $F^{-1}(U)$ $= Pr\{F^{-1}(U) \leq x\} = Pr\{U \leq F(x)\}$ by a.

$= F(x)$

c. If F were to be always a 1-1 function, there would be no need to define F^{-1} in this way & the part b) would have been verified without a)

However, F can be a non 1-1 & we still want b)

Hence we defined the inverse of F & b) holds

PAGE

Problem 3. (5 pts)

i. a. $E(X-a)^2 = EX^2 - 2aEX + a^2 \overset{\text{Let}}{=} f(a)$

Want to minimize $f(a)$

First Order Condition : $\frac{df}{da} = -2EX + 2a \overset{\text{set}}{=} 0$

$\therefore$ Minimum attained at $a = EX$.

(Need to check 2nd order Cond^n & boundary)

$\frac{d^2 f}{da^2} > 0 \qquad \lim_{a \to \pm\infty} f(a) = \infty$

Hence $\min_{a \in \mathbb{R}} E(X-a)^2 = E(X-EX)^2 = Var(X)$

i. b. $E(X-a)^2 = E(X - EX + EX - a)^2$

$= E(X-EX)^2 + \underbrace{(EX-a)^2}_{(*)}$

$2E(X-EX)(EX-a) = 0$

Note $E(X-EX)^2$ does not depend on a

To get minimum, choose $a = EX$ from $(*)$.

Same result with a.

(5 pts)

ii. a. Definition X & Y are uncorrelated.

$\Leftrightarrow \quad cov(X, Y) = 0.$

Recall that $E(X - E(X|Y)) = 0. \Leftrightarrow EX = E(E(X|Y))$

$\therefore \quad cov(X - E(X|Y), E(X|Y)) = E((X-E(X|Y))E(X|Y))$

$= E\left[E\left((X-E(X|Y))E(X|Y) \mid Y\right)\right]$

$= E\left[E(X|Y)\, E\left((X-E(X|Y)) \mid Y\right)\right]$

$= 0$

$\therefore E((X-E(X|Y))|Y)$

$= E(X|Y) - E(X|Y) = 0$

PAGE

b. $Var(X) = E(X - EX)^2$

$= E(X - E(X|Y) + E(X|Y) - EX)^2$

$= \underbrace{E(X - E(X|Y))^2}_{①} + \underbrace{E(E(X|Y) - EX)^2}_{②}$

$\because \underbrace{2E((X - E(X|Y))(E(X|Y) - EX))}_{} = 0$ by part ii.a

$= cov(X - E(X|Y), E(X|Y))$

By definition,

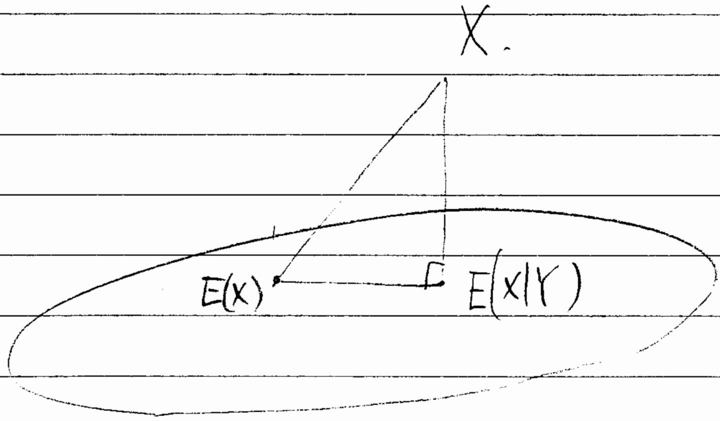
$Var(X|Y) = E((X - E(X|Y))^2 | Y)$

$\therefore E(Var(X|Y)) = E((X - E(X|Y))^2) = ①$

And $② = E(E(X|Y) - EX)^2 = E(E(X|Y) - E(E(X|Y)))^2$

$= Var(E(X|Y))$

(extra 2pts) c.



Inner product $\langle \, , \, \rangle$ is given by $\langle X, Y \rangle = EXY$

& distance $\| \cdot \|$ by $\|X\| = \sqrt{\langle X, X \rangle} = \sqrt{EX^2}$

From a) $X - E(X|Y) \perp E(X|Y) - E(X)$

$Var(X) = E(X - EX)^2$: a squared distance between X & EX

$E((X - E(X|Y))^2)$: " X & $E(X|Y)$

& $E((E(X|Y) - EX)^2)$: " $E(X|Y)$ & EX

Problem 4

(10 pts
2.5 pts each)

A. When $3 \le x \le 4$, $F(x) = \frac{2\cdot 3\cdot x - 3\cdot 4}{24}$

$\therefore F(4) = \frac{12}{24} = \frac{1}{2}$ but $F(4+) = 1$

$\therefore F(x)$ is not right continuous

Hence Not a distribution function.

B. i

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \le x < 2 \\ 1/2 & 2 \le x < 3 \\ 1 & \end{cases}$$

F Satisfies i) $0 \le F(x) \le 1 \quad \forall x$

ii) Nondecreasing

iii) Right Continuous

iv) $F(-\infty) = 0 \quad F(+\infty) = 1$

Hence a distribution function

ii a discrete distribution function. $\because \sum \left[F(x) - F(x-) \right] = 1$.

iv Consider $X = \begin{cases} 1 & \text{w.p } 1/6 \\ 2 & \text{w.p. } 1/3 \\ 3 & \text{w.p. } 1/2 \end{cases}$

Its distribution is exactly same with F.

$\therefore EX = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{2} = \frac{1+4+9}{6} = \frac{7}{3}$

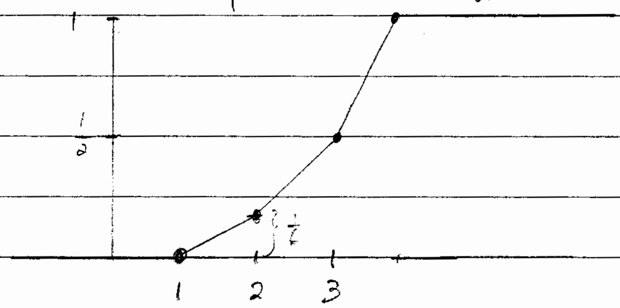
$EX^2 = 1\,\frac{1}{6} + 2^2\cdot\frac{1}{3} + 3^2\,\frac{1}{2} = \frac{1+8+27}{6} = 6$

$Var(X) = 6 - \frac{49}{9} = \frac{5}{9}$

Or $EX = \int_{-\infty}^{\infty} x\, dF(x) = 1\cdot\frac{1}{6} + 2\,\frac{1}{3} + 3\,\frac{1}{2} = \frac{7}{3}$

C.

$$F(x)=\begin{cases} 0 & x<1 \\ (x-1)/6 & 1\le x<2 \\ (2x-3)/6 & 2\le x<3 \\ (x-2)/2 & 3\le x<4 \\ 1 & \text{o.w.} \end{cases}$$



i. Satisfy the required properties as in B.

ii. Continuous $\therefore$ no jump $\Leftrightarrow \sum \left[F(x) - F(x-) \right] = 0$

iv. $EX = \int_{-\infty}^{\infty} x \, dF(x)$ where X is associated w/ F

$$= \int_1^{2-} x \cdot \frac{1}{6}\, dx + \int_2^{3-} x \cdot \frac{1}{3}\, dx + \int_3^{4-} x \cdot \frac{1}{2}\, dx$$

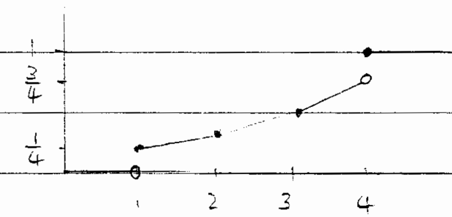
from pdf

$$= \frac{1}{12}(2^2-1) + \frac{1}{6}(3^2-2^2) + \frac{1}{4}(4^2-3^2) = \frac{34}{12} = \frac{17}{6}$$

Likewise $EX^2 = \frac{26}{3}$

$\therefore Var(X) = 0.6389$

D. $F(x) = \frac{1}{2}(\text{Distribution in C}) + \frac{1}{4}$ when $1 \le x < 4$



i. Satisfy the properties as in B. i.e

ii. mixture $\because$ 2 jumps at $x = 1, 4$ $\quad 0 < \sum [F(x) - F(x^-)] < 1$

iii. $F = \frac{1}{2} F_C(x) + \frac{1}{2} F_D(x)$

$F_C(x)$ as in C

$$F_D(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

iv. Consider a r.v $X_D = \begin{cases} 1 & \text{w.p. } 1/2 \\ 4 & \text{w.p } 1/2 \end{cases}$

which corresponds to $F_D(x)$.

Then $EX_D = \frac{5}{2}$ $\quad Var(X_D) = \frac{9}{4}$

Let X be a r.v associated w/ the distribution

Then

$$EX = 1/2 \cdot (\text{Expectation in } C) + 1/2 \cdot EX_D$$

$$= 1/2 \cdot (17/6) + 1/2 \cdot (5/2)$$

$$= 8/3$$

$$Var(X) = 1/2\,(\text{Variance in } C) + 1/2\, Var(X_D)$$

$$+ 1/2 \cdot 1/2 \left(\text{Expectation in } C - EX_D\right)^2$$

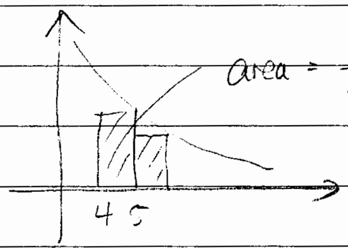
$$= 53/36$$

PAGE

Problem*1 (extra 5pts)

$$P(X=n) = \begin{cases} \dfrac{k}{n^2 \log n \cdot (\log(\log n))^2} & n=5,6,\dots \ \text{for some } k \\ 0 & \text{o.w.} \end{cases}$$

a. Consider $f(x) = \dfrac{k}{x \log x (\log(\log x))^2}$, a decreasing fn of x



$\text{area} = \dfrac{k}{5 \log 5 (\log(\log 5))^2}$

$$EX = \sum_{n=5}^{\infty} n \cdot \frac{k}{n^2 \log n (\log(\log n))^2}$$

$$= \frac{k}{5 \log 5 (\log(\log 5))^2} + \cdots\cdots < \int_4^{\infty} \frac{k}{x \log x (\log(\log x))^2} dx$$

By change of variable $\log(\log x) = y$

$$\frac{1}{\log x} \cdot \frac{1}{x} dx = dy$$

Thus Right Integral

$$= \int_{\log(\log 4)}^{\infty} \frac{k}{y^2} dy$$

$$= \Big[-k y^{-1} \Big]_{\log(\log 4)}^{\infty} = \text{constant} < \infty.$$

b. $E(X^{1+\varepsilon}) = \infty \quad \forall \varepsilon > 0$

For sufficiently large $N(\varepsilon) \in \mathbb{N}$, we can show that

if $n \geq N(\varepsilon)$,

$$\log(\log n) \leq \log n < n^{\varepsilon/3} \quad \text{——} \ (*)$$

First inequality follows from $\log x < x$ if $x > 0$

& $\log(\cdot)$ is an increasing function

For sencond inequality, consider $n < e^{n^{\varepsilon/3}}$

and Taylor expansion of the right hand side.

From $(*)$, $n^{\varepsilon} = n^{\varepsilon/3} \cdot \left(n^{\varepsilon/3}\right)^2 > \log n \cdot \left(\log(\log n)\right)^2$

if $n \geq N(\varepsilon)$

Hence,

$$E(X^{1+\varepsilon}) \geq \sum_{n=N(\varepsilon)}^{\infty} n^{1+\varepsilon} \frac{k}{n^2 \log(n) \left(\log(\log n)\right)^2} > \sum_{n=N(\varepsilon)}^{\infty} \frac{k}{n} = \infty$$