Solution for 171 Homework # 2

1. $\mathcal{U}_\alpha(w) = -\exp\{-\alpha w\} \quad \forall w$

$\lim_{\alpha \to 0} G_\alpha = E(X)$

(i) $\mathcal{U}_\alpha^*(w) = \dfrac{1-\exp\{-\alpha w\}}{\alpha} \quad \forall w$

show $G_\alpha^*(w) = G_\alpha(w)$

$\because \mathcal{U}_\alpha^*(W_0 - G_\alpha^*) = E[\mathcal{U}_\alpha^*(W_0 - X)]$

$$\Rightarrow \frac{1-\exp[-\alpha(W_0 - G_\alpha^*)]}{\alpha} = E\left[\frac{1}{\alpha}[1-\exp(-\alpha(W_0 - X))]\right]$$

$$\Rightarrow \exp(\alpha G_\alpha^*) = E[\exp(\alpha X)] = M_X(\alpha)$$

$$\Rightarrow G_\alpha^* = \frac{\ln M_X(\alpha)}{\alpha}$$

when $\alpha \to 0$ $\lim_{\alpha\to 0} G_\alpha^* = \dfrac{d \ln M_X(\alpha)}{d\alpha} = \dfrac{M_X'(0)}{M_X(0)} = E(X)$

$\therefore G_\alpha^* = G_\alpha$

(ii) $$\lim_{\alpha\to 0} \mathcal{U}_\alpha^*(w) = \lim_{\alpha\to 0} \frac{1-\exp\{-\alpha w\}}{\alpha}$$

$$= \lim_{\alpha\to 0} \frac{(1-\exp\{-\alpha w\})'}{\alpha'}$$

$$= \lim_{\alpha\to 0} w \exp(-\alpha w)$$

$$= w$$

$\therefore$ it is a liner utility function

(iii) $\mathcal{U}(w) = a + bw \qquad b > 0$

$\mathcal{U}(W_0 - G) = E[\mathcal{U}(W_0 - X)]$

$a + b(W_0 - G) = E[a + b(W_0 - X)]$

$a + bW_0 - bG = a + bW_0 - bE(X)$

$\therefore G = E(X)$

1.2 $\quad u(w) = \log w \qquad X = 2^N \qquad f(N) = (\frac{1}{2})^N \quad N=1,2,3,\cdots$

$$E[u(X)] = E[\log 2^N] = \log 2 * E(N)$$

$$E(N) = \sum_{n=1}^{\infty} n(\tfrac{1}{2})^n$$

$$= \sum_{n=1}^{\infty}(\tfrac{1}{2})^n + \sum_{n=2}^{\infty}(\tfrac{1}{2})^n + \sum_{n=3}^{\infty}(\tfrac{1}{2})^n + \cdots$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots$$

$$= 2$$

$$\therefore \; E(u(x)) = 2\log 2$$

1.13 $\quad W_0 = 100 \qquad u(w) = \log w \qquad P_r(X=0) = P_r(X=51) = \frac{1}{2}$

a. $\because \; u(W_0 - G) \geq E[u(W_0 - X)]$

$$\log(100 - G) \geq E[\log(100 - X)]$$

$$E[\log(100 - G)] = \tfrac{1}{2}\log 100 + \tfrac{1}{2}\log(100-51)$$

$$= \tfrac{1}{2}\log 4900$$

$$\Rightarrow 100 - G \geq 70$$

$$\Rightarrow \; G \leq 30$$

b. $\quad u(W_I) = E[u(W_I + H - X)]$

where $u(W_I) = 650$ and $u(w) = \log w$

$$\log 650 = E[\log(650 + H - X)]$$

$$= \tfrac{1}{2}\log(650+H) + \tfrac{1}{2}\log(599+H)$$

$$= \tfrac{1}{2}\log[(650+H)(599+H)]$$

$$\therefore \; 650^2 = (650+H)(599+H)$$

$$\therefore \; H = 26$$

1.14 Complete insurance $G = 40$

$$u(w-G) = u(w-40)$$
$$= -e^{-0.005(w-40)}$$
$$= -1.2214\, e^{-0.005w}$$

Partial insurance $G = 25$

$$E[u(w-G-X+\tfrac{X}{2})]$$
$$= 0.75\, u(w-25) + 0.25\int_0^\infty u(w-25-\tfrac{x}{2}) f(x)\,dx$$
$$= -0.75\, e^{-0.005(w-25)} + 0.25\int_0^\infty -e^{-0.005(w-25-\frac{x}{2})}\, 0.01\, e^{-0.01x}\,dx$$
$$= -1.2276\, e^{-0.005w}$$

$\therefore$ complete insurance maximized the expected utility.

1.18 $f(x) = 0.1e^{-0.1x}$ $x > 0$

a. $X \sim$ exponential ($\lambda = 0.1$)

$$E(X) = \frac{1}{\lambda} = 10$$
$$Var(X) = \frac{1}{\lambda^2} = 100$$

b. pure premium $= 5$

$$E(I(x)) = \beta = 5 = \tfrac{1}{2}E(X)$$
$$I(x) = \frac{x}{2}$$
$$I_d(x) = \begin{cases} 0 & x < d \\ x-d & x \geq d \end{cases}$$

$$\beta = \int_d^\infty [1-F(x)]\,dx$$
$$5 = \int_d^\infty e^{-0.1x}\,dx = 10 * e^{-0.1d}$$
$$d = 10\log 2$$

1.19 $f(x) = \frac{1}{100} \quad 0 < x < 100$

a. $X \sim$ Uniform $(0, 100)$

$E(X) = 50 \qquad Var(X) = \frac{100^2}{12}$

b. $E(I(x)) = \beta = P = 12.5$

$E(kx) = 12.5$

$K = \frac{12.5}{E(X)} = 0.25$ For proportional policy.

For the stop-loss policy, we have

$\beta = \int_d^\infty (1 - F(x))\,dx$

$12.5 = \int_d^{100} (1 - \frac{x}{100})\,dx$

$= \int_d^{100} (1 - \frac{x}{100})\,dx$

$\Rightarrow d = 50$

c. $Var[X - I(x)] = Var[X - 0.25X]$

$= 0.75^2\, Var(X)$

$= 468.75$

$Var[X - I_d(X)] = E[(X - I_d(X))^2] - [E(X - I_d(X))]^2$

$= \int_0^d x^2 \cdot \frac{1}{100}\,dx + \int_d^{100} \frac{d^2}{100}\,dx$

$- \int_0^d x \cdot \frac{1}{100}\,dx - \int_d^{100} \frac{d}{100}\,dx$

$= 260$

$\therefore Var[X - I(x)] > Var[X - I_d(X)]$

1.21 ① $E[(X-I(X)-\mu+\beta)^2]$

$= E[(X-I(X))^2 - 2(\mu-\beta)(X-I(X)) + (\mu-\beta)^2]$

$= E[(X-I(X))^2] - 2(\mu-\beta)^2 + (\mu-\beta)^2$

$= E[(X-I(X))^2] - [E(X-I(X))]^2$

$= Var(X-I(X))$ (here $E(X)=\mu$, $E(I(X))=\beta$)

② $Var(X-I(X))$ is minimized when $I(x) = I_{d^*}(x)$

$\because x^2+z^2 \geq 2xz$ $\therefore x^2-z^2 \geq 2(x-z)z$

$\therefore [x-I(x)]^2 - [x-I_{d^*}(x)]^2 \geq [2x-2I_{d^*}(x)][I_{d^*}(x)-I(x)]$

here if $I_{d^*}(x) = I(x)$ left side = right side = 0

if $I_{d^*}(x) > I(x)$ $I_{d^*}(x) > 0$

$x - I_{d^*}(x) = d^*$

if $I_{d^*}(x) < I(x)$

$I(x) - I_{d^*}(x) \geq 0$ and $I_{d^*}(x) - x \geq -d^*$

$\therefore [2x-2I_{d^*}(x)][I_{d^*}(x)-I(x)] \geq 2[I_{d^*}(x)-I(x)]d^*$

$\therefore E[(X-I(X))^2] - E[(X-I_{d^*}(X))^2] \geq 2E[I_{d^*}(X)-I(X)]d^*$

$= 0$

$\therefore E[(X-I(X))^2] \geq E[(X-I_{d^*}(X))^2]$

$\therefore V[(X-I(X))^2] \geq V[(X-I_{d^*}(X)]$

1.22 From lemma rule

$$u(w) - u(z) \le (w - z)u'(z)$$

for $u''(w) < 0$, $w, z \in [a, b]$

Thus

$$u[w - x + I(x) - P] - u[w - x + I_0(x) - E(x)]$$
$$\le [I(x) - P - I_0(x) + E(x)]\, u'[w - x + I_0(x) - E(x)]$$
$$= [I(x) - P + E(x) - x]\, u'[w - E(x)]$$
$$(\because I_0(x) = x)$$

take expectation

$$E[u(w - x + I(x) - P] - E[u(w - x + I_0(x) - E(x)]$$
$$\le E[I(x) - P + E(x) - x]\, u'[w - E(x)] = 0$$

$$\therefore\ E[u(w - x + I(x) - P)] \le E[u(w - x + I_0(x) - E(x)]$$
$$= E[u(w - E(x))]$$
$$= u(w - E(x))$$

$\therefore$ Full coverage insurance is optimal

1.23 a. $Var[I(x)] = Var[I(x) - X + x]$

$$= Var(X - I(X)) + Var(X) - 2\,cov(X - I(X), X)$$
$$= V + Var(X) - 2\,cov(X - I(X), X)$$

b. $\because$ $\cancel{V + Var(X) \ge 2\,cov(X - I(X), X)}$

$$V + Var(X) \ge 2(V\,Var(X))^{1/2}$$
$$Var[I(X)] \ge 2(V\,Var(X))^{1/2} - 2\,cov(X - I(X), X)$$
$$\because\ Var[I(X)] \ge 0$$

take equal sign.

$$\rho_{X, X - I(X)} = \frac{cov(X, X - I(X))}{(V\,Var(X))^{1/2}} = 1$$

d. $$\rho_{X,\,X-I(X)} = \frac{\mathrm{cov}(X,\,X-I(X))}{(V\,Var(X))^{1/2}} = \frac{\mathrm{cov}(X,\,aX)}{(V\,Var(X))^{1/2}}$$

$$= \frac{a\,Var(X)}{(V\,Var(X))^{1/2}} = 1$$

$$a = \sqrt{V/Var(X)}$$

$$\therefore \quad I(X) = \left[1 - \sqrt{V/Var(X)}\,\right] X$$