

Instructions

- i. This assignment contains some problems in elementary probability to give you an idea of the probabilistic pre-requisites for this course.
- ii. Parts carrying extra credits and starred problems are not required part of the assignment. Nevertheless, a serious attempt followed by a discussion of these during office hours is encouraged.
- iii. Constructive comments on the assignments and for that matter any other aspect of the course will be welcomed.

**Problem 1** In the following  $B(1, 0.5)$  is a symmetric Bernoulli random variable.

1. Let  $X = 2(B(1, 0.5) - 0.5)$  and  $Y = X^2$ .
  - a. Show that  $\text{Cov}(X, Y) = 0$ .
  - b. Show that the two random variables are independent.
2. Let  $X$  be such that

$$P\{X = x\} = \begin{cases} 0.25 & x = -2, -1, 1 \text{ and } 2; \\ 0 & \text{otherwise.} \end{cases}$$

and  $Y = X^2$ .

- a. Show that  $\text{Cov}(X, Y) = 0$ .
- b. Show that the two random variables are dependent.

**Problem 2** This problem explains a method to generate any random variable, be it discrete or continuous, from uniform random variables.

- i. Let  $X$  have an exponential distribution with mean  $\lambda$ . Let  $F$  be its distribution function.
  - a. Find  $F^{-1}(\cdot)$ .
  - b. Show that if  $U$  has a uniform distribution on the interval  $(0, 1)$ , then  $F^{-1}(U)$  has the same distribution as  $X$ .
- ii. Now let  $X$  have a Binomial distribution with parameters  $n = 2$  and  $p = 0.5$ . Let  $F$  be its distribution function. Show that  $F^{-1}(\cdot)$  does not exist.
- iii. To solve the non-existence problem, we define for any distribution function  $F$ , a *generalized inverse*  $F^{-1}(u) = \inf\{z | F(z) \geq u\}$ , where  $\inf$  stands for infimum. If you have not been exposed to infimum, then assume that for the problem it is the same as the minimum. And if you have been exposed, prove that the infimum is attained here for extra credit. Prove that

$$\{u | F^{-1}(u) \leq x\} = (-\infty, F(x)], \quad \forall u \in (0, 1)$$

**Hint:**

$$\begin{aligned} u \leq F(x) &\Rightarrow x \in \{z | F(z) \geq u\} \\ &\Rightarrow x \geq \inf\{z | F(z) \geq u\} \\ &\iff x \geq F^{-1}(u), \quad \forall u \in (0, 1) \end{aligned}$$

- iv. Show that the generalized inverse agrees with the inverse we found in part i.
- v. Find the generalized inverse for the case of the Bernoulli and plot it.
- vi. Let  $U$  be a  $U(0, 1)$  distributed random variable. Show that  $F^{-1}(U)$  has  $F$  as its distribution function.

**Problem 3** In the following assume the existence of all required moments.

i. It is easy to see that

$$\text{Var}(X) = \min_{a \in \mathbb{R}} \mathbb{E}(X - a)^2$$

- a. Prove using Calculus
  - b. Prove without using Calculus
- ii. Let  $X$  and  $Y$  be two random variables such that

$$\Pr(\{X = x, Y = y\}) = \begin{cases} \frac{1}{8} & x = 0, y = 0; \\ \frac{3}{8} & x = 1, y = 0; \\ \frac{1}{4} & x = 0, y = 1; \\ \frac{1}{4} & x = 1, y = 1; \end{cases}$$

Show the following by evaluating the expressions numerically for the above distribution.

a. Show that

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)) \quad \text{and} \quad \mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)).$$

- b. Show that  $X - \mathbb{E}(X|Y)$  is uncorrelated with  $\mathbb{E}(X|Y)$
- c. Show that

$$\text{Var}(X) = \mathbb{E}(\text{Var}(X|Y)) + \text{Var}(\mathbb{E}(X|Y))$$

and

$$\text{Var}(Y) = \mathbb{E}(\text{Var}(Y|X)) + \text{Var}(\mathbb{E}(Y|X))$$

**Problem 4** For the following functions,

- i. Check if they satisfy the required properties of a distribution function.
- ii. If one is found to be a distribution function then classify it as either discrete, continuous or a mixture of a discrete distribution and a continuous distribution.
- iii. In the case that of a mixture of a discrete distribution and a continuous distribution, decompose it into its constituent discrete and continuous components.
- iv. Find the expectation and variance of the random variable associated with the distribution function.

A.

$$F(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1)}{24} & i \leq x \leq i + 1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

B.

$$F(x) = \begin{cases} 0 & x < 1; \\ \frac{i(i+1)}{12} & i \leq x < i + 1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

C.

$$F(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1)}{12} & i \leq x < i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

D.

$$F(x) = \begin{cases} 0 & x < 1; \\ \frac{2ix - i(i+1) + 6}{24} & i \leq x < i+1, \quad i = 1, 2, 3; \\ 1 & \text{otherwise.} \end{cases}$$

**Problem\* 5** Let  $X$  be a random variable taking values in  $\{1, 2, \dots\}$  such that

$$P\{X = n\} \propto \begin{cases} \frac{1}{n^2 \log(n)(\log(\log(n)))^2} & n = 5, 6, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

- Show that  $\mathbb{E}(X) < \infty$
- Show that  $\mathbb{E}(X^{1+\epsilon}) = \infty, \quad \forall \epsilon > 0$