22S:105
Statistical Methods and Computing

Sample size for confidence intervals with σ known
t Intervals

Lecture 13
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The margin of error

- The **margin of error** is the value that we add onto $\bar{x}$ and subtract from $\bar{x}$ to get the endpoints of a confidence interval.
- For confidence intervals for the mean of a normal population with σ known, this is

$$m = z^* \frac{\sigma}{\sqrt{n}}$$

- Equivalently, the margin of error is one half the width of the c.i.
- The margin of error depends on
  - the level of confidence desired
  - the population standard deviation (which we can't control!)
  - the sample size (*not* the population size)

Sample size for a study involving a confidence interval

- Suppose a group of obstetricians wish to carry out a study to estimate μ, the mean birthweight in the population of infants born at UIHC.
- Suppose the obstetricians believe that the population standard deviation of birthweights of infants born at UIHC is the same as that of infants overall in the US.

$$\sigma = 15 \; oz$$

- The obstetricians would like a 95% confidence interval for μ that is no wider than 4 ounces. That is, they want a margin of error ≤ 2 ounce.
- How many infants do they need in their study?

- Let m denote the margin of error. Then

$$\begin{aligned}
m &= z^* \frac{\sigma}{\sqrt{n}} \\
\sqrt{n} &= z^* \; \frac{\sigma}{m} \\
n &= \left(z^* \; \frac{\sigma}{m}\right)^2 \\
n &= \left(1.96 \; * \; \frac{15}{2}\right)^2 \\
&= 216.09
\end{aligned}$$

- A sample size must always be rounded *up*, so they need 217 infants in their study.

Sample size continued

What makes a sample size large?

$$n = \left(z^* \ \frac{\sigma}{m}\right)^2$$

Caveats regarding our formula for computing confidence intervals for population means

- The data must be a *simple random sample* from the population.
  - We are not in too big trouble if the data can plausibly be thought of as observations taken at random from the population.
- "There is no correct method for inference from data haphazardly collected with bias of unknown size. Fancy formulas cannot rescue badly produced data."*
- Watch out for outliers in your dataset, because they can have a large effect on both the point estimate of μ and the confidence interval.

  If outliers are not data errors, and if there is no subject-matter reason for deleting them,

  get help from a statistician on computing measures of center and intervals that are not sensitive to outliers.
- Check your data for skewness and other signs that the population they came from may not be normal. If the sample size is large (i.e. $n \geq 30$) the central limit theorem says the approach is valid. If the sample size is small, the confidence level will not be correct.
- The formula $\bar{x} \ \pm \ z^* \frac{\sigma}{\sqrt{n}}$ requires that we know the exact value of the population standard deviation σ, which we never do.

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* Moore, David S. (2000) *The Basic Practice of Statistics, 2nd ed.*, W.H. Freeman and Co.

What to do when we believe the population is normal but we don't know σ

Assumptions behind this method

- The data are a *simple random sample* from the population of interest.
- Values in the population follow a *normal distribution* with mean μ and standard deviation σ. Both μ and σ are unknown.

The sample mean $\bar{x}$ is still our *point estimate* of the unknown population mean μ.

$\bar{x}$ still comes from a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

- We will *estimate* σ by the sample standard deviation s.
- Then we estimate the standard deviation of $\bar{x}$ by $\frac{s}{\sqrt{n}}$

Standard errors

When we use the data to estimate the standard deviation of a *statistic*, the result is called the *standard error* of the statistic.

The standard error of the sample mean $\bar{x}$ is $\frac{s}{\sqrt{n}}$.

t intervals

When we claimed to know σ, we computed confidence intervals for μ as

$$\bar{x} \ \pm \ z^* \frac{\sigma}{\sqrt{n}}$$

where z^* was the appropriate cutoff value from a standard normal distribution.

When we don't know σ, we will compute confidence intervals for μ as

$$\bar{x} \ \pm \ t^* \frac{s}{\sqrt{n}}$$

When we are *estimating* σ with s, we need to make our confidence interval *wider* to account for the uncertainty in estimation.

- (What if we had gotten a sample that happened to give a sample standard deviation s that was much smaller than the population standard deviation σ?)
- We do this by multiplying $\frac{s}{\sqrt{n}}$ by something *bigger* than z^*.

The t distribution

- There is a different t distribution for every sample size.
  - We identify different t distributions by their *degrees of freedom*, $n - 1$.
- The density curve for t distributions is
  - symmetric around 0
  - bell-shaped (and has only one mode)
- The spread of t distributions is greater than the spread of the standard normal distribution.
  - The smaller the degrees of freedom, the more spread out the t distribution is.
  - The larger the degrees of freedom, the closer the density curve for a t distribution is to a standard normal curve.

- This makes sense because the larger the sample size, the better an estimate s is likely to be for σ (i.e., the less extra uncertainty is introduced by estimating σ instead of knowing its value)

More on the t distribution

If $\bar{x}$ is the sample mean of a simple random sample of size n value from a normal population with mean μ and standard deviation σ, then the random quantity

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

follows a t distribution

Constructing confidence intervals for μ when σ is unknown

To construct a level C confidence interval for μ

- Draw a simple random sample of size n from the population. The population is assumed to be normal.
- Compute the sample mean $\bar{x}$ and the sample standard deviation s.
- Then the level C confidence interval for μ is

  $$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

  where t^* cuts off the upper $\frac{1-C}{2}$ area under the density curve for a t distribution with $n-1$ degrees of freedom.
- Use Table A.2 at the back of your textbook to find t^*.

Example

- We have data on a simple random sample of 10 birthweights of infants born at Boston City Hospital.
- We wish to estimate the mean μ of of birthweights in the population served by this hospital.
- This population may be different from the population of all US birthweights, so we cannot assume that we know either μ or σ.

- Our data values are:

Infant	Birthweight in ounces
1	97
2	117
3	140
4	78
5	99
6	148
7	108
8	135
9	126
10	121

First calculate

$$\bar{x} \ = \ 116.90 \quad s \ = \ 21.70$$

The degrees of freedom are 10 - 1 = 9. For a 95% confidence interval, we need the value of t^* that cuts off an area of .025 in the upper tail.

From Table C, we find $t^* \ = \ 2.262$.

Our confidence interval is

$$\bar{x} \ \pm \ t^* \frac{s}{\sqrt{n}} =$$

$$116.90 \ \pm \ 2.262 \frac{21.70}{\sqrt{10}} =$$

$$116.90 \ \pm \ 15.22 = (101.38, 132.42)$$

The interval is so wide because of

- the relatively small sample size
- the relatively large variation between birthweights (large s)