# 22S:138
# Bayesian Statistics

## Calibration Experiments and Review of Probability

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## Assessing subjective probability about events

- We may sometimes need to quantify our subjective probability of an event in order to make a decision or take an action.
- Example:
  - You have been offered a job as a statistician with a marketing firm in Cincinnati.
  - In order to decide whether to accept the job and move to Cincinnati, you wish to quantify your subjective probability of the event that you would like the job and would like Cincinnati
  - (We will talk about Bayesian decision theory later in the semester.)



## Calibration experiments

- *calibration experiment* — a scale used to assess a person's degree of belief that a particular event will occur (or has occurred)
- All outcomes of the calibration experiment must be *equally likely* in the opinion of the person whose subjective probability is being assessed
  - Example: imagine that I promise to pay you $100 if the roll of a 6-sided die comes up the number you call
  - If you are indifferent as to which number you call, the 6 possible outcomes are equally likely for you



- Calibration experiments may be useful if
  - person is not knowedgeable or comfortable with probability
  - person is uncertain as to his/her opinion about the event
- principle of using a calibration experiment to assess subjective probability
  - Person is offered a choice of 2 ways of winning a prize:
    - through a realization of the calibration experiment with known probability of success
    - through the occurrence of the event of interest.
  - The calibration experiment is adjusted at successive steps

**Example:**

Using a "chips-in-a-bowl" experiment to assess your subjective probability regarding event A: that the Department of Physics at Florida State University has more than 2 female faculty members.

- Experiment is having a blindfolded person draw one chip at random from a bowl containing chips of the same size and shape
- Let $P_s(A)$ denote your subjective probability that event A has occurred.



- Step 1: The bowl contains 1 green chip and 1 red chip.

  Imagine that you may choose 1 of two games.

  1. I will be blindfolded and draw one chip at random. I will pay you \$100 if the chip drawn is red. I will pay you nothing if it is green.
  2. I will pay you \$100 if the Physics Dept at FSU has more than 2 female faculty. I will pay you nothing if it has 2 or fewer.

  If you choose Game 1, then I conclude that

  $$0 \; < P_s(A) \; < \; 0.5$$

  So I construct Step 2 as follows.



- The bowl contains 3 green chips and 1 red chip. You may choose Game 1 or Game 2.

  If you choose Game 2, then I conclude that your

  $$0.25 < P_s(A) < 0.50$$

- Then I may go on to Step 3:

  Now the bowl contains 5 green chips and 3 red chips. You may choose Game 1 or Game 2.

  If you choose Game 1, I conclude that

  $$0.25 < P(A) < 0.375$$



- We continue until it becomes too difficult for you to choose between the two games.
- How the chips are set up in the bowl at each step is determined by your answer at the preceding step.
- Comments:
  - The payoffs have to be imaginary, because we wish to use this procedure for assessing unverifiable probabilities.
  - Luckily, a high degree of accuracy in assessing subjective probabilties usually is not needed.

## Quick review of probability

- **event**: any outcome or set of outcomes of a random phenomenon
  - the basic element to which probability can be applied
  - usual notation is a capital letter near beginning of alphabet
  - example: random phenomenon is that we are drawing a patient at random from a huge database of patients insured by an HMO
    * event A is the event that the patient we draw is under 6 years of age
  - we will denote the probability of event A as $P(A)$
- **sample space S**: the set of all possible outcomes of a random phenomenon
  - $P(S) = 1$



- **intersection** of two events A and B is the event "both A and B"
  - notated $A \cap B$
  - example: if event B is the event that the patient we draw weighs at least 150 pounds, then $A \cap B$ is the event that the patient we draw is under 6 years of age and weighs at least 150 pounds
- **union** of two events is event "either A or B or both"
  - notated $A \cup B$
- A set of events $A_1, A_2, A_3, \ldots$ are **exhaustive** if

$$A_1 \cup A_2 \cup \ A_3 \cup \ldots = S$$



- **complement** of an event $A$ is the event "not A"
  - notated $A^C$ or $\bar{A}$
  - $A \cup \bar{A} = S$
- the **null event** – an event that can never happen
  - notated $\emptyset$
- events $A$ and $B$ are **disjoint** or **mutually exclusive** if they cannot occur together
  - i.e. if $A \cap B = \emptyset$
  - example: event A is that the patient we draw is under 6 years of age, and event B is event that the patient is 6 to 11 years of age



- additive rule of probability
  - if two events $A$ and $B$ are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

- $P(A^C) = 1 - P(A)$

## Conditional Probability

- $P(B|A)$ – the probability that event B will occur given that we already know that event A has occurred
- multiplicative rule of probability

$$P(A \cap B) = P(A)P(B|A)$$
$$P(A \cap B) = P(B)P(A|B)$$

- so if $P(A) \neq 0$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



## Patients in the database example

| | $< 150$ pounds | $\geq 150$ pounds | Total |
|---|---|---|---|
| under 6 | 798 | 2 | 800 |
| $\geq 6$ | 4702 | 4498 | 9200 |
| Total | 5500 | 4500 | 10000 |



## Independence

- Two events are independent if the occurrence (or non-occurrence) of one of them does not affect the probability that the other one occurs. Events $A$ and $B$ are independent if

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

- multiplicative rule of probability for *independent events*

$$P(A \cap B) = P(A)P(B)$$



## Patients in the database example

| | $< 150$ pounds | $\geq 150$ pounds | Total |
|---|---|---|---|
| green eyes | 440 | 360 | 800 |
| not green | 5060 | 4140 | 9200 |
| Total | 5500 | 4500 | 10000 |

## Law of Total Probability

- Applies when you wish to know the marginal (unconditional) probability of some event, but you only know its probability under some conditions.
- Example:
  - I have asked my friend to mail an important letter.
  - I want to calculate $P(A)$, the probability that the letter will reach the addressee within the next 3 days.
  - I believe that $P(M)$, the probability that my friend will remember to mail the letter today or tomorrow, is .60.



  - I believe that if my friend mails the letter today or tomorrow, the probability that the postal service will deliver it to the addressee within the next 3 days is .95.

$$P(A|M) = .95$$

  - I believe there's only 1 chance in 10000 that the letter will get there somehow if my friend forgets to mail it.

$$P(A|\bar{M}) = .0001$$



## Using the Law of Total Probability to find $P(A)$

- For any events $A$ and $M$

$$A = (A \cap M) \cup (A \cap \bar{M})$$

- Events $(A \cap M)$ and $(A \cap \bar{M})$ are disjoint, so the addition rule says

$$P(A) = P(A \cap M) \ + \ P(A \cap \bar{M})$$

- And the multiplication rule applied to both terms on the right hand side says:

$$P(A) = P(A|M)P(M) \ + \ P(A|\bar{M})P(\bar{M})$$

- For the example:

$$\begin{aligned} P(A) &= (.95)(.60) + (0.0001)(.40) \\ &= .57004 \end{aligned}$$



## General Law of Total Probability

- Suppose there were many different conditions under which the event of interest could occur.
- If $M_1, M_2, M_3, \ldots$ are mutually exclusive and exhaustive events then

$$P(A) = P(A|M_1)P(M_1)+P(A|M_2)P(M_2)+P(A|M_3)P(M_3)+\ldots$$

## Bayes' Rule (discrete case)

- My prior probability that my friend would remember to mail the letter was $P(M) = .60$.
- The data is that the letter actually arrived within 3 days!
- Bayes' rule calculates my posterior probability that my friend mailed the letter given the data, that is
$$P(M|A)$$
when we know $P(A|M)$, $P(A|\bar{M})$, and $P(M)$.



- By the definition of conditional probabilility
$$P(M|A) = \frac{P(M \cap A)}{P(A)}$$
- Using the multiplication rule to expand the numerator, and the law of total probability to expand the denominator gives Bayes' rule:
$$P(M|A) = \frac{P(A|M)P(M)}{P(A|M)P(M) + P(A|\bar{M})P(\bar{M})}$$
- For the example this is:
$$\begin{aligned} P(M|A) &= \frac{.95(.60)}{.95(.60) + (0.0001)(.40)} \\ &= \frac{.57}{.57004} \\ &= .99993 \end{aligned}$$



## Generalized Bayes' Rule

- corresponds to Generalized Law of Total Probability
- assumes that the event A could happen conditional on one of a number of different other events, $M_1, M_2, M_3, \ldots$
- assumes that we know $P(A|M_1)$, $P(A|M_2)$, etc. as well as $P(M_1)$, $P(M_2)$, etc.
- after the event A has occurred, we want to assess the conditional probability of one of the events $M_j$, $P(M_j|A)$
- If $M_1, M_2, M_3, \ldots$ are mutually exclusive and exhaustive events then

$$P(M_j|A) = \frac{P(A|M_j)P(M_j)}{P(A|M_1)P(M_1) + P(A|M_2)P(M_2) + P(A|M_3)P(M_3) + \ldots}$$