22S:138
Bayesian Statistics

Inference for Proportions, continued

Lecture 5
Sept. 10, 2007

Kate Cowles
374 SH, 335-0727
kcowles@stat.uiowa.edu

Revisiting the uniform distribution (a noninformative prior for a proportion)

- The uniform distribution is a special case of the Beta distribution.
- What are its parameters?

$$U(0,\ 1) \ = \ Beta(?,\ ?)$$

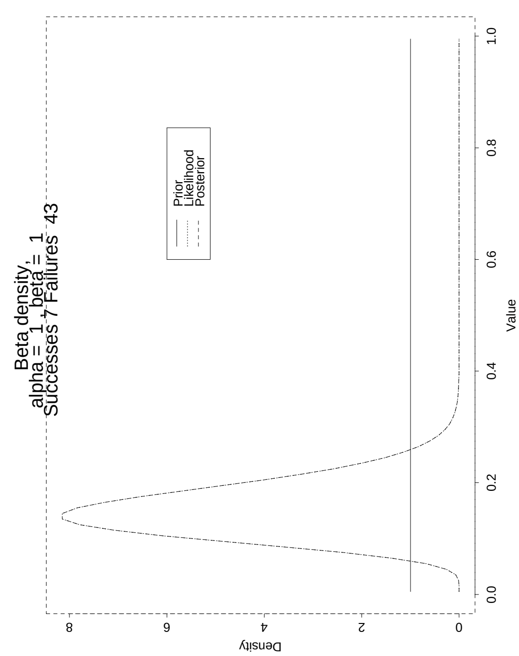
- What is the equivalent prior sample size for a U(0, 1) prior?

  – There is disagreement as to whether the equivalent prior sample size should be defined as
    * $\alpha + \beta$
    * $\alpha + \beta - 1$
    * $\alpha + \beta - 2$

- What is the posterior distribution produced with a U(0, 1) prior and a binomial likelihood?

$$\begin{aligned} p(p|y) \ &= \ Beta(1+y,\ 1+n-y) \\ &\propto \ p^y\ (1-p)^{n-y} \end{aligned}$$

  proportional to the likelihood, as we said before



- Is the posterior mean equal to the MLE $\hat{p}$?

- Note that the *mode* of a $Beta(\alpha,\ \beta)$ distribution is $\frac{\alpha-1}{\alpha+\beta-2}$ So the mode of the posterior distribution given above is $\frac{y}{n} = \hat{p}$.

The posterior distribution contains all the current information about the unknown parameter

All Bayesian inference is based on the posterior distribution.

- estimation
  - estimating values of unknown parameters that can never be observed or known.
- testing
- prediction
  - estimating the values of potentially observable but currently unobserved quantities.
  - e.g., we might want to predict the number of "yesses" in a future survey of 50 UI students

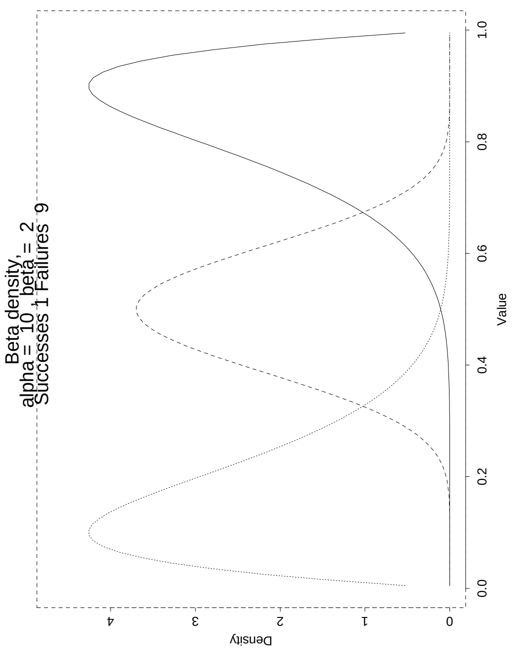
Estimation

- point estimates
- measures of spread
- Bayesian intervals

The posterior variance

- The posterior variance is one summary of the spread of the posterior distr ibution.
- The larger the posterior variance, the more uncertainty we s till have about the parameter.
- See the table of distributions from GCSR for the formula for the variance of a random variable with a beta distribution.
- For a uniform prior and a binomial likelihood, the posterior variance is (almost) always smaller than the prior variance.

When posterior variance is *not* smaller than prior variance



- In our school-quitting example
  - uniform prior
    - prior variance $= \frac{1}{12} = 0.083$
    - posterior variance $= .00246$
  - Beta(10, 40) prior
    - prior variance $= 0.00314$
    - posterior variance $= .00140$

Bayesian intervals

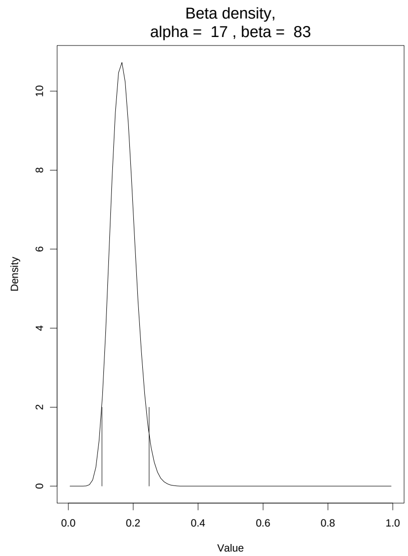
- called "posterior intervals" or "credible sets"
- two kinds
  - equal tail credible sets
  - highest posterior density regions

Equal tail credible sets

- A $100(1-\alpha)\%$ equal tail credible set is the interval from the $\frac{\alpha}{2}$ quantile to the $1-\frac{\alpha}{2}$ quantile of the posterior distribution.
- e.g. if we want a 95% equal tail credible set, α is .05 and we need the .025 and the .975 quantiles.
- We can use built-in Splus functions to get quantiles of standard distributions.

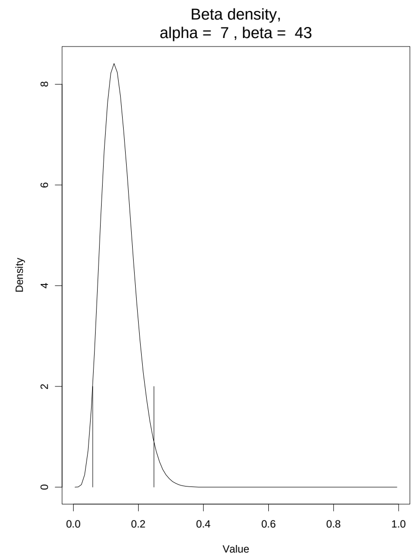
- Example, for our quitting school problem with the Beta(10,40) prior, the posterior was Beta(17, 83).

```
> qbeta( c(0.025, 0.975), 17, 83 )
[1] 0.1033333 0.2491463
```



- If we had instead used a uniform prior, the posterior was Beta(8,44).

```
> qbeta( c(0.025, 0.975), 8,44)
[1] 0.07024083 0.26255154
```



Interpretation of Bayesian intervals

- Recall that the posterior distribution represents our updated subjective probability distribution for the unknown parameter.
- Thus, for us, the interpretation of the 95% credible set is that that the probability is .95 that the true p is in that interval.
- If the Beta(10,40) had been a true representation of our prior beliefs or knowledge about the parameter p, then after seeing our survey data, we would believe that

$$P(0.103 < p < 0.249) = .95$$

- Contrast this with the interpretation of a frequentist confidence interval.

Highest posterior density regions

- the density at any point *inside* the interval is greater than the density at any point outside it
- shortest possible interval trapping the desired probability
- preferable to equal-tail credible sets when posterior is highly skewed or multimodal
- generally difficult to compute; tables of HDRs for certain densities are available

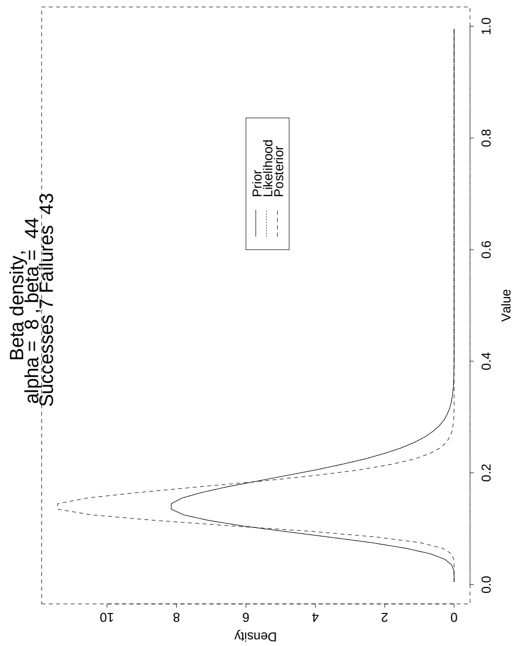
What would go wrong if the new data were used to formulate the prior?

- Worst case:
  - Suppose we know nothing about the problem; our true prior is uniform (ignorance!)
  - suppose we looked at our own survey data and used its normalized likelihood as our prior

$$p(p) \;\propto\; p^7\,(1-p)^{43}, \quad 0 < p < 1$$

- prior would be Beta(8, 44)
- posterior would be Beta(15, 87)
- posterior mean would be $\frac{15}{15+87} = .147$
- posterior variance would be 0.0012
- 95% credible set

```
> qbeta( c(0.025, 0.975), 15, 87 )
[1] 0.08557233 0.22162269
```

Using posterior probabilities to test hypotheses

- Suppose we wanted to test the following hypotheses regarding p:

$$H_0 : p <= .1$$
$$H_1 : \;\; p > .1$$

- We simply need the posterior probabilities of these two ranges of values for p.
- Suppose that the Beta(10, 40) had been our true prior, so our posterior distribution is Beta(17, 83). We can use use a built-in Splus function to obtain $P(p < -.1 \mid y)$.

```
> pbeta(.1, 17, 83)
[1] 0.01879825
```

With this prior, we would conclude that $P(p <= .1 \mid y) = .019$.

- If we instead had used the uniform prior, so our posterior was Beta(8, 44),


```
> pbeta( .1, 8, 44)
[1] 0.1329079
```

With this prior, we would conclude that $P(p <= .1 \mid y) = .133$.

- The interpretation here is totally different from that of a frequentist p-value.

Robustness

- An inference is *robust* if it is not seriously affected by changes in the assumptions on which it is based.
- Assumptions include
  - form of the likelihood
  - parametric family for prior
  - parameters of prior
  - etc.
- Whether an inference is "seriously affected" depends on the purpose of the analysis.
- In this case, if the primary purpose of the analysis was to get a point estimate for p, we might decide that estimation was quite robust to changes in the prior parameters.
- If our primary purpose was the hypothesis test, we might decide otherwise.