22S:138
Bayesian Statistics

Directed Graphs
More on Full Conditionals
Bayesian One-Way Variance Components Model

Lecture 13
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Copying WinBUGS graphs into Microsoft Word for printing

- Left click on plot you wish to copy. A box will appear around the plot indicating that it has been selected.
- Choose "Edit/Copy" from the WinBUGS pull-down menu
- Click in the destination Word document where you want graph to be placed
- Choose "Edit/Paste" from the pull-down menu in Word

Directed graphs

- *directed graphical model* represents all quantities in a statistical model as nodes in a directed graph
- arrows run into nodes from the nodes that directly influence them (i.e. from their *parents*
- developers of BUGS and WinBUGS recommend drawing directed graphs as part of model development process
- WinBUGS includes DoodleBUGS, which lets you specify models as directed graphs instead of using WinBUGS language
  - limited in what kinds of models can be specified in DoodleBUGS

Types of nodes in directed graphs

- constants
  - fixed by design of study
  - always are *founder notes* (i.e. do not have parents
  - denoted as single- or sometimes double-edged rectangles
- stochastic nodes
  - variables that are given a distribution
  - may be parents or children or both
  - may be observed data or unobservable parameters
  - generally denoted as circles
    * athough data often denoted as single-edge rectangles
- deterministic nodes
  - logical functions of other nodes

Types of directed links in a directed graph

- stochastic dependence
  - indicated by solid arrow
- logical function
  - indicated by dashed or hollow arrow

Example: directed graph for Pumps problem

Assumptions behind directed graphical model

- conditional independence assumption: given its parent nodes, each node in the graph is independent of all other nodes except its own "children"
  - given the nodes it is connected to by arrows, any node (say v) is conditionally independent of all the other nodes in the graph
  - collapse over dashed arrows in evaluating connections
- simplifies determination of full conditional distributions

Full conditionals for Pumps problem

Analytical Procedure for extracting full conditionals

- Recall that the full conditional distribution of an unknown model parameter is the distribution of that parameter *given (supposedly known) values of all other quantities in the model*
- Write the mathematical form of the (unnormalized) joint posterior
- Pull out every term in the joint posterior that contains the parameter of interest
- The product of all these terms is proportional to the needed full conditional distribution.
- If possible, identify the parametric family of which the full conditional is a member

Semi-graphical Procedure

- Draw a directed graph of your complete model
- Identify the "parents" and "children" of the parameter of interest
- Write the product of the conditional distributions of
  - the parameter given its parents
  - the children given their parents

One Way Variance-Components Model, Bayesian flavor

- model

$$\begin{aligned} y_{ij}|\alpha_i, \sigma_e^2 &\sim N(\alpha_i, \sigma_e^2) \\ \alpha_i|\mu, \sigma_\alpha^2 &\sim N(\mu, \sigma_\alpha^2) \\ & i = 1, \ldots, K, \quad j = 1, \ldots, n_i \end{aligned}$$

- priors

$$\begin{aligned} \mu &\sim N(\mu_0, \sigma_0^2) \\ \sigma_e^2 &\sim IG(a_1, b_1) \\ \sigma_\alpha^2 &\sim IG(a_2, b_2) \end{aligned}$$

Note: These priors give a very straightforward Gibbs sampling algorithm, but a different prior on σ_α^2 may be preferable. See the new (Sept. 2004) WinBUGS 1.4.2 Dyes example!

- We want posterior means, posterior medians, posterior credible sets for $\alpha, \mu, \sigma_e^2, \sigma_\alpha^2$

Full Conditional Distributions for Variance Components Model

$$\alpha_i | y_{ij}, \mu, \sigma_e^2, \sigma_\alpha^2 \sim$$
$$N(\frac{n_i \sigma_\alpha^2}{n_i \sigma_\alpha^2 + \sigma_e^2} \bar{y}_i + \frac{\sigma_e^2}{n_i \sigma_\alpha^2 + \sigma_e^2} \mu, \frac{\sigma_e^2 \sigma_\alpha^2}{n_i \sigma_\alpha^2 + \sigma_e^2})$$

$$\mu | y_{ij}, \alpha_i, \sigma_e^2, \sigma_\alpha^2 \sim$$
$$N(\frac{K\sigma_0^2}{K\sigma_0^2 + \sigma_\alpha^2} \bar{\alpha} + \frac{\sigma_\alpha^2}{K\sigma_0^2 + \sigma_\alpha^2} \mu_0, \frac{\sigma_0^2 \sigma_\alpha^2}{K\sigma_0^2 + \sigma_\alpha^2})$$

$$\sigma_e^2 | y_{ij}, \alpha_i, \mu, \sigma_e^2, \sigma_\alpha^2 \sim$$
$$IG(a_1 + \frac{\Sigma_{i=1}^K n_i}{2}, b_1 + \frac{\Sigma_{i=1}^K \Sigma_{j=1}^{n_i} (y_{ij} - \alpha_i)^2)}{2})$$

$$\sigma_\alpha^2 | y_{ij}, \alpha_{\mathbf{i}}, \mu, \sigma_{\mathbf{e}}^{\mathbf{2}} \sim$$
$$IG(a_2 + \frac{K}{2}, b_2 + \frac{\Sigma_{i=1}^K (\alpha_i - \mu)^2}{2})$$

Gibbs Sampler algorithm for Variance Components Model

1. choose initial values $\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_K^{(0)}, \mu^{(0)}, \sigma_e^{2(0)}, \sigma_\alpha^2$
2. at each iteration t, generate new value for each parameter, conditional on most recent value of all other parameters

$$\alpha_i^{(t)} | y_{ij}, \mu^{(t-1)}, \sigma_e^{2(t-1}, \sigma_\alpha^{2(t-1)} \sim$$
$$N(\frac{n_i \sigma_\alpha^{2(t-1)}}{n_i \sigma_\alpha^{2(t-1)} + \sigma_e^{2(t-1)}} \bar{y}_i + \frac{\sigma_e^{2(t-1)}}{n_i \sigma_\alpha^{2(t-1)} + \sigma_e^{2(t-1)}} \mu^{(t-}$$
$$\frac{\sigma_e^{2(t-1)} \sigma_\alpha^{2(t-1)}}{n_i \sigma_\alpha^{2(t-1)} + \sigma_e^{2(t-1)}})$$

$$\mu^{(t)} | y_{ij}, \alpha_i^{(t)}, \sigma_e^{2(t-1)}, \sigma_\alpha^{2(t-1)} \sim$$
$$N(\frac{K\sigma_0^{2(t-1)}}{K\sigma_0^2 + \sigma_\alpha^{2(t-1)}} \bar{\alpha}^{(t)} + \frac{\sigma_\alpha^{2(t-1)}}{K\sigma_0^2 + \sigma_\alpha^{2(t)}} \mu_0,$$
$$\frac{\sigma_0^2 \sigma_\alpha^{2(t-1)}}{K\sigma_0^2 + \sigma_\alpha^{2(t-1)}})$$

$$\sigma_e^{2(t)} | y_{ij}, \alpha_i^{(t)}, \mu^{(t)}, \sigma_\alpha^{2(t-1)} \sim$$
$$IG(a_1 + \frac{\Sigma_{i=1}^K n_i}{2}, b_1 + \frac{\Sigma_{i=1}^K \Sigma_{j=1}^{n_i} (y_{ij} - \alpha_i^{(t)})^2)}{2})$$

$$\sigma_\alpha^{2(t)} | y_{ij}, \alpha^{(\mathbf{t})}, \mu^{(\mathbf{t})}, \sigma_{\mathbf{e}}^{\mathbf{2}(\mathbf{t})} \sim$$
$$IG(a_2 + \frac{K}{2}, b_2 + \frac{\Sigma_{i=1}^K (\alpha_i^{(t)} - \mu^{(t)})^2}{2})$$