Let

$$A = \begin{bmatrix} 1 & k_1 & k_1^2 & \cdots & k_1^{n-1} \\ 1 & k_2 & k_2^2 & \cdots & k_2^{n-1} \\ \vdots & \vdots & \vdots & & \\ 1 & k_n & k_n^2 & \cdots & k_n^{n-1} \end{bmatrix}$$

where $k_1, k_2, \ldots, k_n$ are any scalars. A matrix of this form is called a Vandermonde matrix, after Alexandre Theophile Vandermonde (1735-1796). Let us find the determinant of A.

Observe that

$$AT = \begin{bmatrix} 1 & B \\ 1 & 0 \end{bmatrix} \qquad (1)$$

where

$$T = \begin{bmatrix} 1 & -k_n & 0 & \cdots & 0 & 0 \\ 0 & 1 & -k_n & & 0 & 0 \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & -k_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} k_1-k_n & k_1(k_1-k_n) & \cdots & k_1^{n-2}(k_1-k_n) \\ k_2-k_n & k_2(k_2-k_n) & \cdots & k_2^{n-2}(k_2-k_n) \\ \vdots & \vdots & & \vdots \\ k_{n-1}-k_n & k_{n-1}(k_{n-1}-k_n) & \cdots & k_{n-1}^{n-2}(k_{n-1}-k_n) \end{bmatrix}$$

Note that the effect of postmultiplying A by T is to add, to the jth column of A, a scalar multiple of the preceding column ($j = 2,\ldots,n$), thus creating a matrix whose last row is $(1,0,0,\ldots,0)$.

Observe also that B is expressible as

$$B = DC \tag{2}$$

where $D = \mathrm{diag}(k_1-k_n,\ k_2-k_n,\ \ldots,\ k_{n-1}-k_n)$

and

$$C = \begin{bmatrix} 1 & k_1 & k_1^2 & \cdots & k_1^{n-2} \\ 1 & k_2 & k_2^2 & \cdots & k_2^{n-2} \\ \vdots & \vdots & & & \\ 1 & k_{n-1} & k_{n-1}^2 & \cdots & k_{n-1}^{n-2} \end{bmatrix}$$

Thus, B is expressible as a product of a diagonal matrix and of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the last row and column of A. Note that this submatrix (i.e., the matrix C) is an $(n-1) \times (n-1)$ Vandermonde matrix.

Making use of the decompositions (1) and (2) and of basic properties of determinants, we find that

$$\begin{aligned} |A| &= |A||T| = |AT| \\ &= \begin{vmatrix} 1 & B \\ \underline{1} & 0 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} 1 & 0 \\ \underline{1} & B \end{vmatrix} \\ &= (-1)^{n-1}|B| = (-1)^{n-1}|D||C| \\ &= (-1)^{n-1}(k_1-k_n)(k_2-k_n)\cdots(k_{n-1}-k_n)|C| \\ &= (k_n-k_1)(k_n-k_2)\cdots(k_n-k_{n-1})|C| \end{aligned} \tag{3}$$

Formula (3) serves to relate the determinant of an $n \times n$ Vandermonde matrix to that of an $(n-1) \times (n-1)$ Vandermonde matrix, and its repeated application allows us to evaluate the determinant of any Vandermonde matrix.

Clearly, when $n = 2$,

$$|A| = k_2 - k_1 \ ;$$

when $n = 3$

$$|A| = (k_3-k_1)(k_3-k_2)(k_2-k_1) \ ;$$

and. in general,

$$\cdots (k_2-k_1)$$

as can be formally verified by a simple mathematical induction argument based on the relationship (3).

It is evident from formula (4) that $|A| \neq 0$ if and only if $k_j \neq k_i$ for $j > i = 1,\ldots,n$. Thus, A is nonsingular if and only if the n scalars $k_1, k_2, \ldots, k_n$ are distinct.