Sharp Asymptotics for Large Portfolio Losses under Extreme Risks

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Abstract

We study the asymptotic behavior of the loss from defaults of a large portfolio. Inspired by the work of Bassamboo, Juneja, and Zeevi (2008, Operations Research), we consider a static structural model in which latent variables governing individual defaults follow a mixture structure incorporating idiosyncratic risk, systematic risk, and common shock. In our setting, the portfolio effect, namely the decrease in overall risk due to the portfolio size increase, is taken into account by assuming that the individual default thresholds are proportional to a positive deterministic function diverging to infinity. Furthermore, the obligor-specific variables form a sequence of independent and identically distributed vectors, which still allows heterogeneity of the portfolio though. We derive sharp asymptotics for the tail probability of the portfolio loss as the portfolio size becomes large under the assumption, among others, that either the common shock variable or the systematic risk factor has a regularly varying tail. Our main finding is that the occurrence of large losses can be attributed to either the common shock variable or the systematic risk factor, whichever has a heavier tail.

Subject classifications: risk management; portfolio loss; default; sharp asymptotics; common shock; systematic risk; regular variation; law of large numbers.

Area of review: Financial Engineering.

1   Introduction

We are concerned with the loss from defaults of a large credit portfolio of defaultable obligors. As a lesson from the financial crisis of 2007–2009, modeling credit portfolio losses must carefully address extreme risks, which result from the marginal tails of and the tail dependence between individual obligors. It is in general a challenging task to model the intangible tail dependence because it can hardly be perceived under usual economic conditions, but

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suddenly becomes apparent and constitutes a main cause for clustered defaults as the economy deteriorates. In the current state of credit risk management, obligors are assumed to be subject to multi-level risks, roughly categorized as idiosyncratic risk, systematic risk, and common shock. In particular, the common shock symbolizes certain external events, e.g., the collapse of Lehman Brothers, that cause widespread failures and losses of financial institutions and eventually endanger the stability of the financial system. For such cases those Gaussian models become inadequate due to their failure to capture tail dependence resulting from the common shock.

Bassamboo et al. (2008) employ a static structural model for portfolio losses in which each obligor is characterized by a latent variable governing its rating migration and default so that the obligor defaults if the latent variable exceeds a given threshold. Motivated by the multivariate t structure, they propose a mixture structure for the latent variables, which effectively puts idiosyncratic risk, systematic risk, and common shock together and can easily incorporate extremes, extremal dependence, and asymmetry. The individual thresholds are assumed to be proportional to a positive deterministic function diverging to infinity at a subexponential rate in the portfolio size. Under a further assumption, among others, that the common shock variable is regularly varying and dominates the other risk factors, they derive sharp asymptotics, in contrast to existing logarithmic (hence, rough) asymptotics, for the tail probability of portfolio losses. An implication of their result is that large portfolio losses occur primarily due to large values of the common shock variable, while the systematic and idiosyncratic risk factors play a relatively less important role. Nevertheless, as the authors point out, there can be situations where the other risk factors play a dominating role in causing large portfolio losses.

Inspired by the work of Bassamboo et al. (2008), we study the asymptotic behavior of the loss from defaults of a large portfolio. We make some meaningful adjustments and extensions on their model, significantly refine and generalize their theoretical result, and, in particular, complement the asymptotic study by also considering the opposite case that the systematic risk factor is regularly varying and dominates the common shock variable. Our main finding is that the occurrence of large losses can be attributed to either the common shock variable or the systematic risk factor, whichever has a heavier tail. In particular, this implies that, under certain market conditions (e.g., during recessions), the systematic risk inherent in the market may override the exogenous shock to the market in causing credit risk deterioration, which is in sharp contrast to the main finding of Bassamboo et al. (2008).

First, as we target a large credit portfolio, the portfolio effect, namely the decrease in overall risk due to the portfolio size increase, becomes significant and must be appropriately addressed. To feature this, we assume that the individual default thresholds equal $\ell_i f_n$ for $i = 1, \ldots, n$, where f_n is a positive deterministic function diverging to ∞ and ℓ_i's are positive random variables accounting for variations in the portfolio effect on different obligors. Second, we allow those obligor-specific variables, namely the risk exposures θ_i, the idiosyncratic risk factors η_i, and the aforementioned variation factors ℓ_i, to be random and

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assume that they form a sequence of independent and identically distributed (i.i.d.) vectors with a generic copy (θ, η, ℓ). This amounts to identifying a continuum to underlie a large number of obligors of different risk types (hence, a potentially heterogeneous portfolio). For a large portfolio, the cardinality of (θ, η, ℓ) as a continuum follows from the law of large numbers.

The portfolio loss distribution is at the heart of credit risk management, and the asymptotic study has an immediate implication for economic capital assessment, in particular under the current prudent regulatory frameworks. When determining the economic capital requirement, Basel II considers the Asymptotic Single Risk Factor (ASRF) model and stipulates that the economic capital is estimated to guarantee the solvency of the bank over a one-year horizon at a 99.9% confidence level.[1] In practice, many banks select an even more conservative confidence level between 99.96% and 99.98% in economic capital models.[2] Essentially, in assessing the economic capital, high-level quantiles of the portfolio loss are of considerable interest, for which case our sharp asymptotic estimates become powerful.

We end this section with a brief literature review on the asymptotic study of large portfolio losses. For conditionally independent credit risk models, either static or dynamic, either structural or reduced-form, a usual procedure in the literature is to first condition on common risk factors, then employ standard approaches in limit theory including the law of large numbers (LLN), the central limit theorem (CLT), and the large deviation principle (LDP), and finally integrate out the conditioning risk factors.

Among early works on this topic, Vasicek (1987, 1991) studies the loss of a large homogeneous loan portfolio, implicitly assumes a Gaussian copula between different borrowers, and derives a simple closed-form limiting distribution for the loss. Lucas et al. (2001) and Gordy (2003) study the loss distribution of a large heterogeneous portfolio and obtain LLN-type limiting distributions in the form of conditional expectation given common factors. The former work also examines the portfolio size required to render the asymptotic loss distribution a good approximation to the actual loss distribution. Dembo et al. (2004) employ the LDP approach to derive a precise approximation for the portfolio loss of a static model in which the exposures and defaults are independent conditional on a macro-environmental variable. Schloegl and O'Kane (2005) extend Vasicek's work to the case of t copula, derive closed-form solutions for the portfolio loss distribution, and compare the Value-at-Risk implied by the t copula to that by the Gaussian, Clayton, and Gumbel copulas. Glasserman et al. (2007) analyze the tail behavior of the portfolio loss in the Gaussian copula setting and establish logarithmic limits (hence, rough estimates) for the tail of the loss distribution in two limiting regimes, namely the small default probability regime and the large portfolio loss threshold regime. Pra et al. (2009) consider large homogeneous portfolios within the

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[1]See, e.g., An Explanatory Note on the Basel II IRB Risk Weight Functions released in 2005 by the Bank for International Settlements (BIS) available at `https://www.bis.org/bcbs/irbriskweight.htm`.

[2]See, e.g., Economic Capital and the Assessment of Capital Adequacy released in 2004 by the Federal Deposit Insurance Corporation (FDIC) available at `https://www.fdic.gov/regulations/examinations/supervisory/insights/siwin04/siwinter04-article1.pdf`.

class of reduced-form models based on interacting intensities, and they conduct a comprehensive asymptotic study of portfolio losses by employing all of the LLN, CLT, and LDP approaches. Bush et al. (2011) consider a large portfolio in a structural model under a dynamic setting and study the loss function of the portfolio through a stochastic partial differential equation. Cvitanić et al. (2012) and Giesecke et al. (2013, 2015) consider large portfolios of interacting obligors in reduced-form models with a self-exciting common factor, and establish LLN-type results for portfolio losses.

As the portfolio size increases, the decrease in overall risk makes large portfolio losses become rarer and, hence, more difficult to observe under the naive Monte Carlo method. In this regard, importance sampling becomes a commonly used alternative to increase the efficiency of simulation. A desired importance sampling algorithm is the one under which the importance sampling estimator possesses either bounded relative error or asymptotic optimality. Asymptotic tail estimates for portfolio losses can usually serve as a key input in constructing an importance sampling distribution to fulfill the requirement. For general introductions to importance sampling and rare-event simulation, see Heidelberger (1995), Juneja and Shahabuddin (2006), and Asmussen and Glynn (2007). For applications of importance sampling to large portfolio losses, see Glasserman and Li (2005), Bassamboo et al. (2008), Glasserman et al. (2008), Chan and Kroese (2010), Brereton et al. (2012), and Liu (2015), among others.

The rest of the paper is organized as follows. Section 2 elaborates on the modeling of large portfolio losses. Section 3 prepares some preliminaries including a primary observation on a simplified case where individual default thresholds do not vary with the portfolio size. Section 4 exhibits the main results for the cases of a regularly varying common shock variable or a regularly varying systematic risk factor. Section 5 conducts some numerical studies to check the accuracy of the obtained formulas and compute the Value-at-Risk of the portfolio loss at high levels. Section 6 makes some concluding remarks. All proofs are relegated to Appendix.

2   Modeling large portfolio losses

Consider a large credit portfolio of n defaultable obligors. For each obligor i, we introduce a latent variable Z_i that summarizes the determinants of the obligor's rating migration and default. Denote by x_i^n the individual default threshold of obligor i, which can be exogenously given and related to the portfolio size n. Similarly to Bassamboo et al. (2008), assume that

$$x_i^n = \ell_i f_n, \qquad i = 1, \ldots, n,$$

where f_n is a positive deterministic function diverging to ∞ as $n \to \infty$ and each variation factor ℓ_i is a positive random variable.

As explained by Bassamboo et al. (2008), introducing a diverging function f_n to individual default thresholds ensures that the probability of large portfolio losses diminishes as

n increases, which is true for low-default portfolios. Moreover, such a specification of individual default thresholds allows us to account for the portfolio effect, namely the decrease in overall risk as the portfolio size increases. To explain this, assume that as the portfolio expands each latent variable Z_i is modified to $\frac{Z_i}{\iota_i f_n}$, where the positive diverging function f_n is used to reflect an overall improvement on the credit quality, while a positive random variable ι_i to reflect a minor variation in portfolio effect on obligor i. Denote by $a_i > 0$ the endogenously determined default threshold of obligor i, and write $\ell_i = a_i \iota_i$. In this way, obligor i defaults if and only if $\frac{Z_i}{\iota_i f_n} > a_i$ if and only if $Z_i > \ell_i f_n$.

Confined to this static structural model, the loss given default of obligor i is described as

$$\theta_i 1_{(Z_i > \ell_i f_n)},$$

where θ_i is a positive random variable denoting the risk exposure at default and 1_A is the indicator function of an event A. Such a loss model descends from Merton's firm-value model and has been commonly used in the literature. The exposure at default appearing above is another key parameter in modeling portfolio losses; see Hon and Bellotti (2016), Leow and Crook (2016), and Tong et al. (2016) for recent discussions on its randomness.

We follow the work of Bassamboo et al. (2008) to employ a mixture model for the underlying latent variables:

$$Z_i = S\left(\rho\xi + \sqrt{1-\rho^2}\eta_i\right), \qquad i = 1, \ldots, n, \tag{2.1}$$

where each η_i is a real-valued random variable interpreted as an idiosyncratic risk factor that affects obligor i only, ξ is a real-valued random variable interpreted as a systematic risk factor inherent in the entire market, S is a positive random variable to capture a common shock, while $0 < \rho < 1$ is a coefficient to adjust the roles of the systematic and idiosyncratic risk factors. Thus, this is a conditionally independent model in the sense that defaults of obligors conditional on S and ξ are independent, consistent with many existing works in credit risk modeling. See Frey and McNeil (2003) and McNeil et al. (2015) for related discussions and for a number of conditionally independent models that are in spirit similar to (2.1). The well-known Gaussian and t models can easily be retrieved from the mixture model (2.1) by suitably specifying the distributions of these risk factors. Moreover, it is straightforward to extend this model to the case of a vector $\boldsymbol{\xi}$ so as to accommodate multiple systematic risk factors; see, e.g., Kostadinov (2005), who conducts research under the multivariate elliptical framework.

In the mixture model (2.1), the common shock variable S refers to a stylized representation of unpredictable changes in certain exogenous factors that create an economy-wide shock on all obligors. This is closely related to the concept of systemic risk, which has become a hot topic of paramount importance for financial stability in the post financial crisis era. For related discussions on this topic, see Elsinger et al. (2006), Tarashev et al. (2010), Giesecke and Kim (2011), Ang and Longstaff (2013), and Acharya et al. (2017), some of whom argue coexistence of systematic risk and systemic risk under extreme market

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conditions. Now that the common shock variable S is introduced as a mixing variable in (2.1), in certain circumstances it may represent a main driving force for systemic risk.

Collectively, the portfolio loss from defaults is modeled as

$$L_n = \sum_{i=1}^{n} \theta_i 1_{\left(S\left(\rho\xi + \sqrt{1-\rho^2}\eta_i\right) > \ell_i f_n\right)}. \tag{2.2}$$

In this model, each individual loss contains five variables, θ_i, S, ξ, η_i, and ℓ_i, among which those indexed by i are obligor-specific variables while the other two impact on the whole portfolio. We assume that $(\theta_i, \eta_i, \ell_i)$, $i = 1, \ldots, n$, form a sequence of i.i.d. random vectors with a generic copy (θ, η, ℓ), and that $\{(\theta_i, \eta_i, \ell_i), i = 1, \ldots, n\}$, S, and ξ are mutually independent. Subsequently, the latent variables $Z_1, \ldots, Z_n$ are identical to

$$Z = S\left(\rho\xi + \sqrt{1-\rho^2}\eta\right).$$

Note that among the five random variables, θ, S, and ℓ take values from $\mathbb{R}_+ = (0, \infty)$ while ξ and η take values from $\mathbb{R} = (-\infty, \infty)$.

In their Assumption 1, Bassamboo et al. (2008) assume that the sequence $\{(\theta_i, \ell_i), i = 1, \ldots, n\}$ is deterministic and takes values in a finite set of elements and that the proportion of each element, namely, the ratio of the number of pairs (θ_i, ℓ_i) equal to the element over the portfolio size n, converges to a positive number. This amounts to assuming that (θ_i, ℓ_i), $i = 1, 2, \ldots$, are i.i.d. random pairs with a common distribution over a finite set (hence, a special case of ours). A similar assumption is made by Glasserman et al. (2007). Since we consider a large portfolio, our idea is to introduce a continuum of obligors, which naturally follows from the LLN. Consider a special case that the cardinality of (θ, η, ℓ), namely the support set of its joint distribution, is a finite set. This becomes similar to, but is still more general than, the one considered by Bassamboo et al. (2008). Such a special case can be interpreted as a heterogeneous credit portfolio consisting of a finite number of homogeneous sub-portfolios, each comprising obligors of the same risk type. Therefore, even under the i.i.d. assumption we are dealing with a large, potentially heterogeneous, portfolio.

It is important to note that the risk embodied in the idiosyncratic risk factors is subject to the diversification effect, but not the risk embodied in the systematic risk factor and the common shock variable, as examined by Sicking et al. (2018) in a numerical study. This motivates us to take conditional expectation given S and ξ in establishing our main results. Following this procedure, however, the impact of the idiosyncratic risk factors turns out to be neglected.

Our main results are sharp asymptotics for the tail probability of the portfolio loss L_n as $n \to \infty$. For an arbitrarily fixed number $b \in (0, E\theta)$, under the assumption that the common shock variable S has a regularly varying tail $\overline{F_S} = 1 - F_S$ dominating that of the systematic risk factor ξ, we establish a sharp asymptotic formula

$$P(L_n > nb) = (C_1 + o(1))\,\overline{F_S}(f_n),$$

while under the assumption that ξ has a regularly varying tail $\overline{F_\xi}$ dominating that of S, we establish another sharp asymptotic formula

$$P\left(L_n > nb\right) = \left(C_2 + o(1)\right) \overline{F_\xi}(f_n),$$

where C_1 and C_2 are two positive constants expressed in explicit forms, and each $o(1)$ stands for a function of n which tends to 0 as $n \to \infty$; see Theorems 4.1 and 4.2 below. These results offer a new insight that the occurrence of large losses is determined by whichever one of S and ξ has a heavier tail. Intuitively, in the mixture model (2.1), the idiosyncratic risk factor η_i vanishes when applying the LLN, leaving the common shock S and the systematic risk factor ξ to roughly play a symmetric role.

3 Preliminaries

3.1 Notational conventions

Throughout the paper, all limit relations without specifications are according to $n \to \infty$. For any $x, y \in \mathbb{R}$, write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $x_+ = x \vee 0$. Denote by F_X the distribution of a random variable X, by $F_{X,Y}$ the joint distribution of a random vector (X, Y), and so on, letting the notation speak for itself. For two positive functions g_1 and g_2, we write $g_1 \sim g_2$ if $\lim g_1/g_2 = 1$, write $g_1 \lesssim g_2$ or $g_2 \gtrsim g_1$ if $\limsup g_1/g_2 \le 1$, write $g_1 = o(g_2)$ if $\lim g_1/g_2 = 0$, and write $g_1 = O(g_2)$ if $\limsup g_1/g_2 < \infty$. For a real function g, we denote its left and right limits at x by $g(x-)$ and $g(x+)$, respectively, which exist if g is monotone. For a non-decreasing function $g : \mathbb{R} \to \mathbb{R}$, denote by $g^\leftarrow$ and $g^\rightarrow$ its càglàd and càdlàg inverses; that is, for $y \in \mathbb{R}$,

$$\begin{cases} g^\leftarrow(y) = \inf\{x \in \mathbb{R} : g(x) \ge y\}, \\ g^\rightarrow(y) = \sup\{x \in \mathbb{R} : g(x) \le y\}, \end{cases}$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$ by convention. In particular, if the level y corresponds to the value of x at which $g(\cdot)$ strictly increases, then the two inverses $g^\leftarrow$ and $g^\rightarrow$ coincide, for which case we record this unique inverse as $g^\leftarrow$.

3.2 A primary observation

To gain some hints for the study, we explore a simplified case with $f_n \equiv f > 0$ being fixed. Due to the independence between the sequence $\{(\theta_i, \eta_i, \ell_i), i = 1, \ldots, n\}$ and the vector (S, ξ), by conditioning on (S, ξ) we write

$$P\left(L_n > nb\right) = \iint_{\mathbb{R}_+ \times \mathbb{R}} P\left(\frac{L_n}{n} > b \,\middle|\, S = s, \xi = t\right) P\left(S \in ds, \xi \in dt\right). \tag{3.1}$$

Under $P\left(\cdot \,|\, S = s, \xi = t\right)$, by the LLN it holds that, almost surely,

$$\frac{L_n}{n} \quad = \quad \frac{1}{n} \sum_{i=1}^n \theta_i \mathbf{1}_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta_i\right) > \ell_i f\right)}$$

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$$\to \quad E\left[\theta 1_{\left(s\left(\rho t+\sqrt{1-\rho^2}\eta\right)>\ell f\right)}\right]$$
$$= \quad r_0(s,t), \tag{3.2}$$

where we assume $E\theta < \infty$. Thus, for any $b \in (0, E\theta)$, it holds for arbitrarily fixed small $\varepsilon, \delta > 0$ and all large n, say, $n \geq n_0(\varepsilon, \delta, s, t)$, that

$$1_{(r_0(s,t)>b+\delta)} - \varepsilon \leq P\left(\frac{L_n}{n} > b \,\middle|\, S = s, \xi = t\right) \leq 1_{(r_0(s,t)>b-\delta)} + \varepsilon.$$

Applying Fatou's lemma to the right-hand side of (3.1), we obtain

$$\iint_{r_0(s,t)>b+\delta} P\left(S \in ds, \xi \in dt\right) - \varepsilon \lesssim P\left(L_n > nb\right) \lesssim \iint_{r_0(s,t)>b-\delta} P\left(S \in ds, \xi \in dt\right) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, it follows that

$$\iint_{r_0(s,t)>b} P\left(S \in ds, \xi \in dt\right) \lesssim P\left(L_n > nb\right) \lesssim \iint_{r_0(s,t)\geq b} P\left(S \in ds, \xi \in dt\right). \tag{3.3}$$

It is noteworthy that the derivation above does not require S and ξ to be independent.

Clearly, if $P\left(r_0(S,\xi) = b\right) = 0$, then both bounds in (3.3) coincide, yielding a precise limit

$$\lim_{n\to\infty} P\left(L_n > nb\right) = \iint_{r_0(s,t)>b} P\left(S \in ds, \xi \in dt\right). \tag{3.4}$$

This condition holds under some additional mild assumptions. For this purpose, first assume that (S, ξ) is jointly continuously distributed. Next, note that, since θ is strictly positive, the function $r_0(s,t)$ exhibits exactly the same positivity, continuity, and monotonicity as the probability function

$$p_0(s,t) = P\left(s\left(\rho t + \sqrt{1-\rho^2}\eta\right) > \ell f\right). \tag{3.5}$$

Define the set

$$D_0 = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : 0 < r_0(s,t) < E\theta\} = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : 0 < p_0(s,t) < 1\},$$

where the second equality is still due to the strict positivity of θ. If $D_0 = \emptyset$, which happens in case both η and ℓ are degenerate, then $P\left(r_0(S,\xi) = b\right) = 0$ automatically holds. Now consider $D_0 \neq \emptyset$, for which case D_0 does not reduce to a singleton due to the left-continuity of $p_0(s,t)$ in both s and t. Obviously, $p_0(s,t)$ is non-decreasing in both $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$. We strengthen it to that $p_0(s,t)$ strictly increases over the set D_0 in the sense that $p_0(s_1,t_1) < p_0(s_2,t_2)$ for all (s_1,t_1) and (s_2,t_2) from D_0 with $s_1 < s_2$ and $t_1 < t_2$, and so does $r_0(s,t)$. This guarantees that $r_0(s,t) = b$ does not allow a rectangle for (s,t), and thus $P\left(r_0(S,\xi) = b\right) = 0$.

We conclude the following:

Proposition 3.1 *Consider the portfolio loss (2.2) with $f_n \equiv f > 0$ fixed and $E\theta < \infty$. Assume that (S, ξ) is jointly continuously distributed and that the function $p_0(s, t)$ defined by (3.5) strictly increases over the set D_0 when $D_0 \neq \emptyset$. Then relation (3.4) holds for any fixed $b \in (0, E\theta)$, where the function $r_0(s, t)$ is defined by (3.2).*

In this paper, however, we consider the case that f_n diverges to ∞, which, as explained before, is to reflect the rarity of large losses or to account for the portfolio effect. For this case, relation (3.4) becomes trivial since $\frac{L_n}{n}$ under $P(\cdot \mid S = s, \xi = t)$ converges to 0 almost surely and, hence, the set $(r_0(s, t) > b)$ is empty. In order to capture the sharp asymptotic behavior of the tail probability $P(L_n > nb)$, we need to assume that either $\overline{F_S}$ or $\overline{F_\xi}$ is regularly varying.

3.3 Regular variation

Recall that a positive function g on $\mathbb{R}_+$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $g \in \mathrm{RV}_\alpha$, if

$$\lim_{x \to \infty} \frac{g(xy)}{g(x)} = y^\alpha, \qquad y > 0.$$

When $\alpha = 0$, this defines a slowly varying function at ∞. See Bingham et al. (1987) and Resnick (1987) for textbook treatments of regular variation. Consider a real-valued random variable X having a regularly varying tail $\overline{F_X} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$. Then $\overline{F_X}(x)$ is a power-like function in the sense that it differs from the power function $x^{-\alpha}$ by up to a slowly varying function at ∞. The regular variation of $\overline{F_X}$ can be restated as follows: There exists a Radon measure ν non-degenerate on $\mathbb{R}_+$ such that

$$\lim_{x \to \infty} \frac{P\left(\frac{X}{x} \in A\right)}{\overline{F_X}(x)} = \nu(A) \tag{3.6}$$

holds for every interval $A \subset \mathbb{R}_+$ away from 0. This measure ν is actually given by

$$\nu(s, \infty) = s^{-\alpha} \qquad \text{for } s > 0. \tag{3.7}$$

As a typical example, let X be a random variable following the Pareto distribution of type I,

$$F_X(x) = 1 - \left(\frac{c}{x}\right)^\alpha, \qquad x > c, \tag{3.8}$$

with parameters $\alpha, c > 0$ (hence, $\overline{F_X} \in \mathrm{RV}_{-\alpha}$). Then we have

$$\frac{P\left(\frac{X}{x} \in ds\right)}{\overline{F_X}(x)} = \nu(ds) \qquad \text{over } s > \frac{c}{x}, \tag{3.9}$$

which is consistent with relation (3.6).

4 Main results

4.1 Under a regularly varying common shock variable

Let us first conduct a heuristic analysis on a special case that the common shock variable S follows the Pareto distribution (3.8), and then rigorize the finding by establishing a theorem. Motivated by the primary observation in Subsection 3.2, we expand the probability $P(L_n > nb)$ for $b \in (0, E\theta)$ by conditioning on $\left(\frac{S}{f_n}, \xi\right)$, where S is scaled by f_n to make $\frac{L_n}{n}$ conditionally have a proper limit. Precisely,

$$P(L_n > nb) = \iint_{\mathbb{R}_+ \times \mathbb{R}} P\left(\frac{L_n}{n} > b \,\Big|\, \frac{S}{f_n} = s, \xi = t\right) P\left(\frac{S}{f_n} \in ds\right) P(\xi \in dt).$$

Then by (3.9), we have

$$\frac{P(L_n > nb)}{\overline{F_S}(f_n)} = \iint_{\left(\frac{c}{f_n}, \infty\right) \times \mathbb{R}} P\left(\frac{L_n}{n} > b \,\Big|\, \frac{S}{f_n} = s, \xi = t\right) \nu(ds) P(\xi \in dt). \tag{4.1}$$

For any $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$, the conditional expectation of an individual loss is

$$r_1(s, t) = E\left[\theta 1_{(Z > \ell f_n)} \,\Big|\, \frac{S}{f_n} = s, \xi = t\right] = E\left[\theta 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta\right) > \ell\right)}\right], \tag{4.2}$$

which takes values in $[0, E\theta]$ and is non-decreasing in both $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$. Similarly to the derivation in Subsection 3.2, under $P\left(\cdot \,\Big|\, \frac{S}{f_n} = s, \xi = t\right)$, by the LLN it holds that, almost surely,

$$\frac{L_n}{n} = \frac{1}{n} \sum_{i=1}^{n} \theta_i 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta_i\right) > \ell_i\right)} \to r_1(s, t).$$

This urges us to replace $\frac{L_n}{n}$ by $r_1(s, t)$ in (4.1). Then following the analysis in Subsection 3.2, it is plausible to obtain that

$$\iint_{r_1(s,t) > b} \nu(ds) P(\xi \in dt) \lesssim \frac{P(L_n > nb)}{\overline{F_S}(f_n)} \lesssim \iint_{r_1(s,t) \geq b} \nu(ds) P(\xi \in dt). \tag{4.3}$$

We remark that this heuristics can be validated by applying Fatou's lemma subject to a suitable moment condition on ξ, but we omit details here since we are going to establish a more general result and give it a rigorous proof.

Clearly, if

$$\iint_{r_1(s,t) = b} \nu(ds) P(\xi \in dt) = 0, \tag{4.4}$$

then both bounds in (4.3) coincide, yielding a precise limit relation

$$\lim_{n \to \infty} \frac{P(L_n > nb)}{\overline{F_S}(f_n)} = \iint_{r_1(s,t) > b} \nu(ds) P(\xi \in dt). \tag{4.5}$$

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For this purpose, note that, as in Subsection 3.2, the function $r_1(s,t)$ exhibits exactly the same positivity, continuity, and monotonicity as the probability function

$$p_1(s,t) = P\left(s\left(\rho t + \sqrt{1-\rho^2}\eta\right) > \ell\right). \tag{4.6}$$

Then define the set

$$D_1 = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : 0 < r_1(s,t) < E\theta\} = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : 0 < p_1(s,t) < 1\}.$$

If $D_1 = \emptyset$, which happens in case both η and ℓ are degenerate, then $r_1(s,t) = b$ defines an empty set for (s,t) and thus relation (4.4) holds. Now consider $D_1 \neq \emptyset$. Since $\nu(ds)$ defined by (3.7) is continuous over $\mathbb{R}_+$, for relation (4.4) to hold, it suffices to assume that $r_1(s,t)$, or equivalently $p_1(s,t)$, strictly increases in s over the set D_1. For this purpose, Lemma A.1 below shows several sufficient conditions on $F_{\eta,\ell}$, which essentially encompass all cases of (η, ℓ) of practical interest.

Below is our first main result in which we consider the case of a general regularly varying tail $\overline{F_S}$ and show that, upon some technical conditions, relation (4.5) is indeed valid.

Theorem 4.1 *Consider the portfolio loss (2.2) and assume the following:*
- *$\overline{F_S} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$;*
- *$E\xi_+^\beta < \infty$ for some $\beta > \alpha$;*
- *$x\overline{F_\theta}(x) = o\left(\overline{F_S}(x)\right)$ as $x \to \infty$ (hence, $E\theta < \infty$);*
- *$p_1(s,t)$ defined by (4.6) strictly increases in s over the set D_1 when $D_1 \neq \emptyset$.*

Then relation (4.5), with the right-hand side finite, holds for any fixed $b \in (0, E\theta)$ and $f_n = O(n)$.

The first two conditions together mean that the common shock variable has a heavier tail than the systematic risk factor; in other words, they describe the situation that, as during the financial crisis of 2007–2009, the exogenous shock to the market overrides the systematic risk inherent in the market. Thus, our Theorem 4.1 offers the same insight as Theorem 1 of Bassamboo et al. (2008) into the different roles of the three risk sources. Precisely, the tail behavior of the portfolio loss L_n is approximated by that of the common shock variable S, while the systematic and idiosyncratic risk factors ξ and η_i contribute to the prefactor in the approximation only. It is noteworthy that, in order to gain this insight, Bassamboo et al. (2008) assume, among others, that both $\overline{F_\xi}$ and $\overline{F_\eta}$ are bounded by an exponentially decaying term and that η possesses a probability density function. In establishing our Theorem 4.1, such technical assumptions are largely avoided or significantly weakened.

Example 4.1 Assume that F_η has a support set to be a finite or infinite interval and that ℓ is degenerate at a positive constant l, so that according to Lemma A.1(a) the last condition of Theorem 4.1 is fulfilled. This can also be verified directly by observing the function

$$r_1(s,t) = E\left[\theta 1_{\left(\rho t + \sqrt{1-\rho^2}\eta > \frac{l}{s}\right)}\right], \qquad (s,t) \in \mathbb{R}_+ \times \mathbb{R}.$$

In particular, if further θ and η are independent then

$$r_1(s,t) = E\theta \cdot \overline{F_\eta}\left(\frac{\frac{l}{s} - \rho t}{\sqrt{1-\rho^2}}\right).$$

We are going to convert the double integral on the right-hand side of (4.5) into an iterated integral. Define

$$\tilde{r}_1(t) = r_1(\infty, t) = E\theta \cdot \overline{F_\eta}\left(-\frac{\rho t}{\sqrt{1-\rho^2}}\right);$$

see also relation (A.3) below. Then for any $b \in (0, E\theta)$, the function $\tilde{r}_1(t)$ has a unique inverse

$$t(b) = -\frac{\sqrt{1-\rho^2}}{\rho} F_\eta^{\leftarrow}\left(1 - \frac{b}{E\theta}\right),$$

where $F_\eta^{\leftarrow}$ denotes the unique inverse of F_η. Furthermore, for any $b \in (0, E\theta)$ and $t > t(b)$, the function $r_1(\cdot, t)$ has a unique inverse as

$$s_t(b) = l\left(\rho t + \sqrt{1-\rho^2} F_\eta^{\leftarrow}\left(1 - \frac{b}{E\theta}\right)\right)^{-1}.$$

To convert the double integral on the right-hand side of (4.5) into an iterated integral, further assume that ξ is continuously distributed. Then using (3.7) we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \frac{P(L_n > nb)}{\overline{F_S}(f_n)} &= \int_{t(b)}^{\infty} \int_{s_t(b)}^{\infty} \nu(ds) P(\xi \in dt) \\
&= \frac{1}{l^\alpha} \int_{-\frac{\sqrt{1-\rho^2}}{\rho} F_\eta^{\leftarrow}\left(1 - \frac{b}{E\theta}\right)}^{\infty} \left(\rho t + \sqrt{1-\rho^2} F_\eta^{\leftarrow}\left(1 - \frac{b}{E\theta}\right)\right)^\alpha P(\xi \in dt).
\end{aligned}
$$

4.2 Under a regularly varying systematic risk factor

Similarly to Subsection 4.1, let us first conduct a heuristic analysis on another special case that the systematic risk factor ξ follows the Pareto distribution (3.8), and then rigorize the finding by establishing another theorem. In this case, we expand the probability $P(L_n > nb)$ for $b \in (0, E\theta)$ by conditioning on $\left(S, \frac{\xi}{f_n}\right)$ instead, where the purpose of scaling ξ by f_n is still to make $\frac{L_n}{n}$ conditionally have a proper limit. Precisely,

$$P(L_n > nb) = \iint_{\mathbb{R}_+ \times \mathbb{R}} P\left(\frac{L_n}{n} > b \;\middle|\; S = s, \frac{\xi}{f_n} = t\right) P(S \in ds) P\left(\frac{\xi}{f_n} \in dt\right).$$

Then by (3.9), we have

$$\frac{P(L_n > nb)}{\overline{F_\xi}(f_n)} = \iint_{\mathbb{R}_+ \times \left(\frac{c}{f_n}, \infty\right)} P\left(\frac{L_n}{n} > b \;\middle|\; S = s, \frac{\xi}{f_n} = t\right) P(S \in ds) \nu(dt).$$

We are going to derive upper and lower bounds of the conditional probability above. For this purpose, define the function

$$r_2(u) = E\left[\theta 1_{(\ell < \rho u)}\right], \qquad u \in \mathbb{R}_+, \tag{4.7}$$

which is left continuous. Under $P\left(\cdot \,\middle|\, S = s, \frac{\xi}{f_n} = t\right)$, by the LLN, it holds for any $M > 0$ and all large n that, almost surely,

$$\begin{aligned}
\frac{L_n}{n} &= \frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\,\frac{\eta_i}{f_n}\right) > \ell_i\right)} \\
&\leq \frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\,\frac{|\eta_i|}{M}\right) > \ell_i\right)} \\
&\to E\left[\theta 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\,\frac{|\eta|}{M}\right) > \ell\right)}\right] \\
&\downarrow \quad r_2(st+) \qquad \text{as } M \uparrow \infty,
\end{aligned}$$

and

$$\begin{aligned}
\frac{L_n}{n} &\geq \frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t - \sqrt{1-\rho^2}\,\frac{|\eta_i|}{M}\right) > \ell_i\right)} \\
&\to E\left[\theta 1_{\left(s\left(\rho t - \sqrt{1-\rho^2}\,\frac{|\eta|}{M}\right) > \ell\right)}\right] \\
&\uparrow \quad r_2(st) \qquad \text{as } M \uparrow \infty.
\end{aligned}$$

Following the analysis in Subsection 3.2, it is plausible to obtain that

$$\iint_{r_2(st)>b} P\left(S \in ds\right)\nu(dt) \lesssim \frac{P\left(L_n > nb\right)}{\overline{F_\xi}(f_n)} \lesssim \iint_{r_2(st+)\geq b} P\left(S \in ds\right)\nu(dt). \tag{4.8}$$

Indeed, subject to a suitable moment condition on S, this heuristics can be validated by applying Fatou's lemma, but we omit details here since we are going to establish a more general result and give it a rigorous proof.

To pursue equality of both bounds in (4.8), define the set

$$D_2 = \{u \in \mathbb{R}_+ : 0 < r_2(u) < E\theta\}.$$

If $D_2 = \emptyset$, which happens in case ℓ is degenerate at $l > 0$, say, then the restrictions $r_2(st) > b$ and $r_2(st+) \geq b$ in (4.8) are equivalent to $\rho st > l$ and $\rho st \geq l$, respectively. Hence it follows from (4.8) that

$$\lim_{n\to\infty} \frac{P\left(L_n > nb\right)}{\overline{F_\xi}(f_n)} = \iint_{\rho st \geq l} P\left(S \in ds\right)\nu(dt) = \left(\frac{\rho}{l}\right)^\alpha ES^\alpha, \tag{4.9}$$

where the last step is due to (3.7). Now consider $D_2 \neq \emptyset$. Note that, as in Subsections 3.2 and 4.1, the function $r_2(u)$ exhibits the same positivity, continuity, and monotonicity as the probability function $P(\ell < \rho u)$. Also note that

$$D_2 = \{u \in \mathbb{R}_+ : 0 < P(\ell < \rho u) < 1\} = \frac{1}{\rho}\{u \in \mathbb{R}_+ : 0 < F_\ell(u-) < 1\}.$$

Assume that $F_\ell(u)$ strictly increases in u over the set $\{u \in \mathbb{R}_+ : 0 < F_\ell(u) < 1\}$, so does $F_\ell(u-)$ over the set $\{u \in \mathbb{R}_+ : 0 < F_\ell(u-) < 1\}$. Then $P(\ell < \rho u)$ and, hence, $r_2(u)$ strictly increase in u over D_2. Thus for any $b \in (0, E\theta)$, the function $r_2(u)$ has a unique inverse $r_2^\leftarrow(b)$. By the left continuity and strict monotonicity of $r_2(u)$ over D_2, it is easy to check the following: for $b \in (0, E\theta)$,

$$r_2(u) > b \Longleftrightarrow u > r_2^\leftarrow(b), \qquad r_2(u+) \geq b \Longleftrightarrow u \geq r_2^\leftarrow(b).$$

Then rewrite the regions of the two double integrals in (4.8) and convert each double integral into an iterated integral. Using (3.7) we obtain

$$\lim_{n\to\infty} \frac{P(L_n > nb)}{\overline{F_\xi}(f_n)} = \frac{ES^\alpha}{(r_2^\leftarrow(b))^\alpha}. \tag{4.10}$$

By the way, this result actually allows a degenerate ℓ as a special case because for this case relation (4.10) reduces to relation (4.9).

The following is our second main result in which we consider the case of a general regularly varying tail $\overline{F_\xi}$ and show that, upon some technical conditions, relation (4.10) is indeed valid.

Theorem 4.2 *Consider the portfolio loss (2.2) and assume the following:*

- *$\overline{F_\xi} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$;*
- *$ES^\beta < \infty$ for some $\beta > \alpha$;*
- *$x\overline{F_\theta}(x) = o\left(\overline{F_\xi}(x)\right)$ as $x \to \infty$ (hence, $E\theta < \infty$);*
- *$F_\ell(u)$ strictly increases in u over the set $\{u \in \mathbb{R}_+ : 0 < F_\ell(u) < 1\}$ if it is nonempty.*

Then relation (4.10) holds for any fixed $b \in (0, E\theta)$ and $f_n = O(n)$.

The first two conditions together mean that the systematic risk factor has a heavier tail than the common shock variable; in other words, they describe the situation that, as during recessions, the systematic risk inherent in the market overrides the exogenous shock to the market. Compared to Theorem 4.1, Theorem 4.2 shows that the tail behavior of the portfolio loss L_n is approximated by that of the systematic risk factor ξ, while the common shock variable S contributes to the prefactor in the approximation only and the idiosyncratic risk factors η_i even completely disappear. This finding is in sharp contrast to that of Bassamboo et al. (2008).

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Example 4.2 Assume $\ell = le^{U_\sigma}$, where l is a positive constant and U_σ is a random variable independent of θ and uniformly distributed over $(-\sigma, \sigma)$ for some $\sigma > 0$. We have

$$r_2(u) = E\theta \cdot \left[\left(\frac{\log\left(\frac{\rho u}{l}\right) + \sigma}{2\sigma} \right)_+ \wedge 1 \right], \qquad u \in \mathbb{R}_+,$$

which strictly increases from 0 to $E\theta$ as u increases from $\frac{l}{\rho} e^{-\sigma}$ to $\frac{l}{\rho} e^{\sigma}$. Thus, for any $b \in (0, E\theta)$, its unique inverse is

$$r_2^\leftarrow(b) = \frac{l}{\rho} \exp\left\{ 2\sigma \frac{b}{E\theta} - \sigma \right\},$$

and relation (4.10) becomes

$$\lim_{n \to \infty} \frac{P\left(L_n > nb\right)}{\overline{F_\xi}(f_n)} = \left(\frac{\rho}{l}\right)^\alpha \exp\left\{ \alpha\sigma - 2\alpha\sigma \frac{b}{E\theta} \right\} ES^\alpha.$$

By the way, letting $\sigma \downarrow 0$, which leads to the case that ℓ is degenerate at l, the relation above is further simplified to (4.9).

5 Numerical studies

In this section, we perform numerical studies to check the accuracy of approximations given by formulas (4.5) and (4.10) by Monte Carlo simulation and conduct a sensitivity analysis on the Value-at-Risk of the portfolio loss to key model parameters including the adjusting coefficient ρ and the regular variation index α.

For simplicity, obligor-specific variables θ, η, and ℓ are assumed to be mutually independent, though this is not required by the two theorems. Moreover, θ is specified to be an exponential random variable with mean 800, the portfolio size n varies from 10 to 1000, and the sample size for the simulation is set to $N = 10^8$.

5.1 A numerical study of Theorem 4.1

We first check the accuracy of the approximation given by formula (4.5) under a regularly varying common shock variable S. Other model specifications for this numerical study are listed below:

- S follows a Pareto distribution of type II, with tail

$$\overline{F_S}(s) = \left(\frac{1}{s+1}\right)^{1.5}, \qquad s > 0,$$

  so that $\overline{F_S} \in \mathrm{RV}_{-\alpha}$ with $\alpha = 1.5$;
- ξ and η are i.i.d. normal variables with mean 2 and variance 1;

- $\rho = 0.6$;
- ℓ follows a three-point distribution, with $P(\ell = 2) = 0.1$, $P(\ell = 2.75) = 0.5$, and $P(\ell = 3.5) = 0.4$;
- $f_n = 10 + n^{0.4}$.

Under these specifications, the individual default probability when the portfolio size n equals 10, 100, 1000 is computed to be 2.0%, 1.4%, 0.7%, respectively, each of which indicates a low-default credit portfolio. It is easy to check that all conditions required by Theorem 4.1 are fulfilled. For example, since $\mathrm{supp}(F_{\eta,\ell}) = \mathbb{R} \times \{2, 2.75, 3.5\}$, by Lemma A.1(b), the last condition of Theorem 4.1 holds.

Recall the function $r_1(s,t)$ defined in (4.2) and its limit function $\tilde{r}_1(t)$ defined by (A.3). Similarly to Example 4.1, we convert the double integral on the right-hand side of (4.5) into an iterated integral as

$$\lim_{n\to\infty} \frac{P\left(L_n > nb\right)}{\overline{F_S}(f_n)} = \int_{t(b)}^{\infty} (s_t(b))^{-\alpha} P(\xi \in dt), \tag{5.1}$$

where $t(b)$ with $b \in (0, E\theta)$ given denotes the unique solution to the equation $\tilde{r}_1(t) = b$, and $s_t(b)$ with $b \in (0, E\theta)$ and $t \in \mathbb{R}$ given denotes the unique solution to the equation $r_1(s,t) = b$. The existence and uniqueness of the solutions to these equations can easily be verified under the current model specifications.

Figure 5.1 compares the simulated $\frac{P(L_n > nb)}{\overline{F_S}(f_n)}$ with the limit given by (5.1) on the left and shows their ratio on the right, where b varies from 50 to 750. We observe that the simulated values converge to the limit as n increases, and that when $n = 1000$ the approximation error is less than 5%. Although Theorem 4.1 claims that the limit relation (4.5) holds for any $b \in (0, 800)$, Figure 5.1 shows that the approximation error increases when b approaches 0 or 800. The poor performance when b approaches 0 should be due to the rarity of the event $(L_n \leq nb)$, while the poor performance when b approaches 800 should be due to the rarity of the event $(L_n > nb)$. For these two extreme scenarios, special treatments such as deriving second-order asymptotics may help improve the quality of the approximation.

Value-at-Risk (VaR) is one of the primary risk measures employed by financial institutions to determine the amount of economic capital for unexpected losses. We conduct a sensitivity analysis on the VaR of the portfolio loss L_n estimated from formula (5.1) with respect to the adjusting coefficient ρ and the regular variation index α. By (5.1), for a given confidence level $q \in (0,1)$, $\mathrm{VaR}_q(L_n) = n\mathrm{VaR}_q\left(\frac{L_n}{n}\right)$ can be approximated by

$$\widehat{\mathrm{VaR}}_q(L_n) = nb^*(q),$$

where $b^*(q)$ denotes the unique solution to the equation

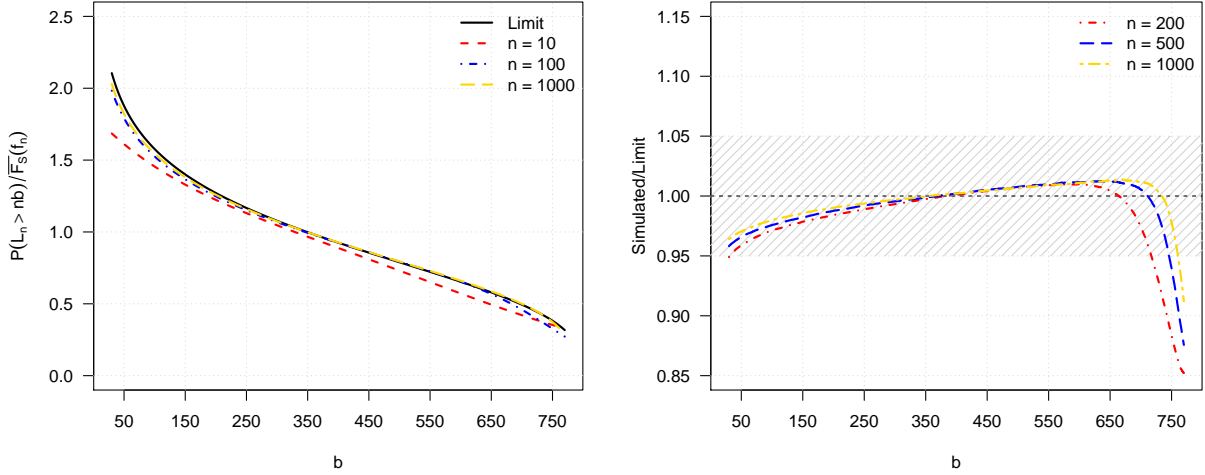$$\overline{F_S}(f_n) \int_{t(b)}^{\infty} (s_t(b))^{-\alpha} P(\xi \in dt) = 1 - q. \tag{5.2}$$

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Figure 5.1: Comparison between the simulated $\frac{P(L_n > nb)}{\overline{F_S}(f_n)}$ and its limit given by Theorem 4.1.

To see the existence and uniqueness of the solution to equation (5.2), we observe that, under the model specifications above, the right-hand side of (4.5) is continuous and strictly decreasing in b, and so is the left-hand side of equation (5.2).

For a fixed portfolio size $n = 1000$, Table 5.1 summarizes percentage changes in $\widehat{\text{VaR}}_q(L_n)$ with respect to percentage changes in ρ and α, for $q = 99.4\%$, 99.5%, and 99.6%, respectively. It shows that $\widehat{\text{VaR}}_q(L_n)$ increases as ρ increases, which is anticipated because a higher value of ρ means more systematic risk the portfolio is exposed to, and hence higher likelihood of simultaneous defaults. It also shows that $\widehat{\text{VaR}}_q(L_n)$ increases when α decreases, which is also anticipated because a smaller value of α means a heavier tail of the common shock variable, and hence higher likelihood of simultaneous defaults. We observe that $\widehat{\text{VaR}}_q(L_n)$ is much more sensitive to α than to ρ, which is due to the dominance of the common shock variable S over the whole portfolio. Moreover, as the confidence level q increases, which indicates that the financial institution becomes more prudent, the sensitivity of $\widehat{\text{VaR}}_q(L_n)$ to α decreases noticeably, while the sensitivity to ρ almost remains unchanged.

5.2 A numerical study of Theorem 4.2

We first check the accuracy of the approximation given by formula (4.10) under a regularly varying systematic risk factor ξ. Other model specifications for this numerical study are listed below:

- S follows a Gamma$(2, 1)$ distribution with density se^{-s} for $s > 0$;
- ξ and η are i.i.d. following a common Pareto distribution of type II, with tail

$$\overline{F_\xi}(t) = \left(\frac{1}{t+1} \right)^{1.6}, \qquad t > 0,$$

so that $\overline{F_\xi} \in \text{RV}_{-\alpha}$ with $\alpha = 1.6$;

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Model parameters		$\widehat{\text{VaR}}_q(L_n)$		
		$q = 99.4\%$	$q = 99.5\%$	$q = 99.6\%$
% change in ρ	$+2\%$	$+0.6\%$	$+0.6\%$	$+0.5\%$
	$+1\%$	$+0.3\%$	$+0.3\%$	$+0.3\%$
	($\rho = 0.6$)	(4.66×10^5)	(5.70×10^5)	(6.64×10^5)
	-1%	-0.3%	-0.3%	-0.3%
	-2%	-0.7%	-0.6%	-0.6%
% change in α	$+2\%$	-13.6%	-9.8%	-6.1%
	$+1\%$	-6.8%	-4.8%	-2.9%
	($\alpha = 1.5$)	(4.66×10^5)	(5.70×10^5)	(6.64×10^5)
	-1%	$+6.7\%$	$+4.5\%$	$+2.7\%$
	-2%	$+13.2\%$	$+8.8\%$	$+5.0\%$

Table 5.1: Sensitivity testing for VaR of L_n on ρ and α using Theorem 4.1.

- $\rho = 0.85$;
- $\ell = 0.5 + 6B$, where B follows a Beta$(0.9, 3)$ distribution with density

$$2.4795x^{-0.1}(1-x)^2, \qquad 0 < x < 1;$$

- $f_n = 10 \log n$.

Under these specifications, the individual default probability when the portfolio size n equals 10, 100, 1000 is computed to be 2.0%, 0.7%, 0.4%, respectively, which reflects a more significant portfolio effect than in Subsection 5.1. It is easy to check that all conditions required by Theorem 4.2 are fulfilled.

The accuracy of the approximation given by formula (4.10) is examined in Figure 5.2. Similarly to the previous numerical study in Subsection 5.1, for $n = 1000$ and for intermediate values of b, the simulated values for $\frac{P(L_n > nb)}{\overline{F}_\xi(f_n)}$ are within 3% of the limit, but as b approaches 0 or 800 the performance of the estimation becomes poor.

Again, we conduct a sensitivity analysis on the VaR of the portfolio loss L_n estimated from formula (4.10) with respect to ρ and α. By (4.10), for a given confidence level $q \in (0, 1)$, we have the approximation

$$\widehat{\text{VaR}}_q(L_n) = nb^*(q),$$

where $b^*(q)$ denotes the unique solution to the equation

$$\frac{\overline{F}_\xi(f_n)ES^\alpha}{(r_2^\leftarrow(b))^\alpha} = 1 - q.$$

Under the model specifications above, the solution $b^*(q)$ assumes an analytical expression

$$b^*(q) = E\theta \cdot F_\ell\left(\rho\left(\frac{\overline{F}_\xi(f_n)ES^\alpha}{1-q}\right)^{\frac{1}{\alpha}}\right).$$
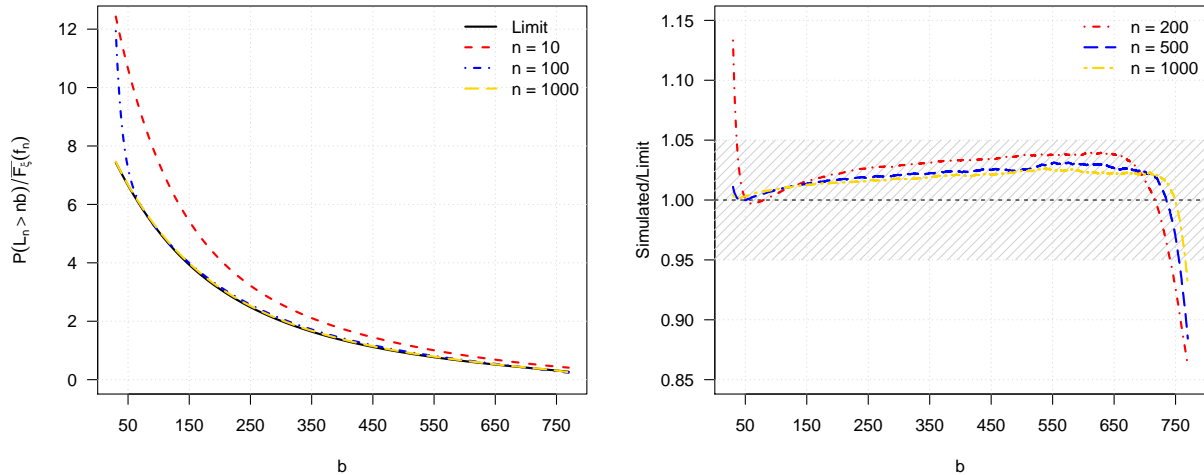
Figure 5.2: Comparison between the simulated $\frac{P(L_n > nb)}{\overline{F}_\xi(f_n)}$ and its limit given by Theorem 4.2.

The same as the previous numerical study in Subsection 5.1, for a fixed portfolio size $n = 1000$, Table 5.2 summarizes percentage changes in $\widehat{\text{VaR}}_q(L_n)$ with respect to percentage changes in ρ and α, for $q = 99.4\%$, 99.5%, and 99.6%, respectively. Similarly, it shows that $\widehat{\text{VaR}}_q(L_n)$ increases as ρ increases or as α decreases. However, the sensitivity of $\widehat{\text{VaR}}_q(L_n)$ to ρ, though still much less than that to α, becomes much more significant than in Table 5.1. This is anticipated because in the current situation the systematic risk factor ξ plays a more dominating role in the whole portfolio. Moreover, as the confidence level q increases, the sensitivity of $\widehat{\text{VaR}}_q(L_n)$ to both ρ and α decreases noticeably.

Model parameters		$\widehat{\text{VaR}}_q(L_n)$		
		$q = 99.4\%$	$q = 99.5\%$	$q = 99.6\%$
% change in ρ	$+2\%$	$+6.7\%$	$+5.1\%$	$+3.9\%$
	$+1\%$	$+3.4\%$	$+2.5\%$	$+2.0\%$
	$(\rho = 0.85)$	(0.89×10^5)	(1.24×10^5)	(1.69×10^5)
	-1%	-3.4%	-2.5%	-2.0%
	-2%	-6.8%	-5.1%	-4.0%
% change in α	$+2\%$	-18.8%	-14.7%	-12.0%
	$+1\%$	-9.6%	-7.5%	-6.1%
	$(\alpha = 1.6)$	(0.89×10^5)	(1.24×10^5)	(1.69×10^5)
	-1%	$+9.9\%$	$+7.7\%$	$+6.3\%$
	-2%	$+20.1\%$	$+15.8\%$	$+12.9\%$

Table 5.2: Sensitivity testing for VaR of L_n on ρ and α using Theorem 4.2.

6 Concluding remarks

We study the loss from defaults of a large, potentially heterogeneous, portfolio in a static structural model in which the latent variables governing individual defaults follow a mixture structure, the portfolio effect is taken into account, and the obligor-specific variables constitute a continuum. We derive sharp asymptotics for the tail probability of the portfolio loss, showing that the occurrence of large losses can be attributed to either the common shock variable or the systematic risk factor, whichever has a heavier tail.

Several extensions of our work are worthy of pursuit in the future. First, in our work we follow the usual procedure to condition on the common shock variable and the systematic risk factor and then employ the LLN approach. In doing so, the impact of the idiosyncratic risk factors turns out to be neglected. However, there can be situations where idiosyncratic risk plays a dominating role in causing large portfolio losses; see, e.g., Ang and Longstaff (2013). Thus, it will be interesting to extend the asymptotic study to such situations and capture the impact of idiosyncratic risk factors. Second, it is highly desirable to establish CLT and LDP-type approximations for large portfolio losses in various situations, as such approximations are anticipated to be more accurate than the ones obtained through the LLN approach. Moreover, they may give a clue on how to capture the impact of idiosyncratic risk factors and, hence, answer the first question. Third, the use of the indicator function in the portfolio loss model (2.2) indicates that once an obligor defaults its loss rate is 100%, which is impractical. To remedy this, we can follow Shi et al. (2017) to introduce a non-decreasing function taking values in $[0, 1]$ to link the loss rate to the severity of default. Actually, there is a vast literature on modeling the loss rate; see, e.g., Calabrese and Zenga (2010), Calabrese (2014), Yao et al. (2015, 2017), Betz et al. (2018), and Hurlin et al. (2018). Even more realistically, we can follow this literature to model the exposure at default, the loss rate, and the default in an integrated way that all of them share some common risk factors and each has its own idiosyncratic risk factor. Finally, confined to the mixture structure, the common shock, systematic risk, and idiosyncratic risk play different roles in causing credit risk propagation and deterioration. By virtue of the asymptotic study, it is possible to quantitatively analyze and distinguish the different roles of these risk factors. Such a quantitative analysis is fascinating and has important implications for credit risk management.

Appendix   Proofs of the main results

Lemmas

We firstly construct technical conditions to guarantee the monotonicity of the function $p_1(s, t)$ defined in (4.6), as required in proving that the two bounds in (4.3) are equal.

Recall that the support set of a random vector $\mathbf{X}$ or its distribution function $F_{\mathbf{X}}$, denoted by $\mathrm{supp}(F_{\mathbf{X}})$, is the closure of the set of all possible values of $\mathbf{X}$. In other words, $\mathbf{x} \in \mathrm{supp}(F_{\mathbf{X}})$ if and only if $P(\mathbf{X} \in \Delta(\mathbf{x})) > 0$ for any neighborhood of $\mathbf{x}$. It turns out to be crucial for our purpose to assume that a support set is connected; that is, it cannot be partitioned into two nonempty subsets such that each subset has no points in common with the closure of the other. It is easy to see that $F_{\mathbf{X}}$ is strictly increasing in every dimension over the interior of $\mathrm{supp}(F_{\mathbf{X}})$ if the interior is nonempty, but not necessarily continuous even if $\mathrm{supp}(F_{\mathbf{X}})$ is connected.

Recall the probability function $p_1(s, t)$ defined in (4.6):

$$p_1(s, t) = P\left(s\left(\rho t + \sqrt{1 - \rho^2}\eta\right) > \ell\right), \qquad (s, t) \in \mathbb{R}_+ \times \mathbb{R}.$$

We restrict it to the set $D_1 = \{(s, t) \in \mathbb{R}_+ \times \mathbb{R} : 0 < p_1(s, t) < 1\}$.

Lemma A.1 *The probability function $p_1(s, t)$ strictly increases in s over the set D_1 under either of the following conditions:*

(a) $\mathrm{supp}(F_{\eta,\ell})$ *is connected;*
(b) $\mathrm{supp}(F_{\eta,\ell}) = \mathbb{R} \times \mathrm{supp}(F_\ell)$;
(c) $\mathrm{supp}(F_{\eta,\ell}) = \mathrm{supp}(F_\eta) \times \mathbb{R}_+$.

Proof. Fix $t \in \mathbb{R}$ and denote by $\tilde{D}_1$ the cross section of the set D_1 at t. Necessarily,

$$P\left(\rho t + \sqrt{1 - \rho^2}\eta > 0\right) > 0 \tag{A.1}$$

because otherwise $\tilde{D}_1 = \emptyset$. Then we need to prove that $p_1(s, t)$ strictly increases in $s \in \tilde{D}_1$. Define

$$Y = \frac{1}{\ell}\left(\rho t + \sqrt{1 - \rho^2}\eta\right)$$

and denote by y_* and y^* the infimum and supremum, respectively, of

$$\mathrm{supp}(F_Y) = \left\{\frac{1}{l}\left(\rho t + \sqrt{1 - \rho^2}u\right) : (u, l) \in \mathrm{supp}(F_{\eta,\ell})\right\}.$$

Since $p_1(s,t) = P(sY > 1)$, it suffices to prove that F_Y strictly increases at $y = \frac{1}{s}$ for every $s \in \tilde{D}_1$. Furthermore, for every $s \in \tilde{D}_1$, since $(s,t) \in D$ we have $0 < P\left(Y > \frac{1}{s}\right) = p_1(s,t) < 1$, which implies that $y_* \leq \frac{1}{s} \leq y^*$. Thus, for a fixed $y \in (y_* \vee 0, y^*)$, it suffices to prove that F_Y strictly increases at y. We are going to prove this in the three cases.

(a) Since $\text{supp}(F_{\eta,\ell})$ is connected and $y \in (y_*, y^*)$, we can always find $(u,l) \in \text{supp}(F_{\eta,\ell})$ such that $y = \frac{1}{l}\left(\rho t + \sqrt{1-\rho^2}u\right)$. Thus,

$$(Y \in dy) \supset (\eta \in du, \ell \in dl), \tag{A.2}$$

which precisely means that, for any neighborhood $\Delta(y)$ of y, we can find a neighborhood $\Delta(u,l)$ of (u,l) such that $(Y \in \Delta(y)) \supset ((\eta,\ell) \in \Delta(u,l))$. Thus, F_Y strictly increases at y.

(b) Arbitrarily choose $l \in \text{supp}(F_\ell)$ and then let $u = \frac{ly-\rho t}{\sqrt{1-\rho^2}} \in \mathbb{R}$, so that (A.2) still holds. Since $\text{supp}(F_{\eta,\ell}) = \mathbb{R} \times \text{supp}(F_\ell)$, we have $(u,l) \in \text{supp}(F_{\eta,\ell})$ and hence F_Y strictly increases at y.

(c) By (A.1), we can find $u \in \text{supp}(F_\eta)$ such that $\rho t + \sqrt{1-\rho^2}u > 0$. Then let $l = \frac{\rho t + \sqrt{1-\rho^2}u}{y} \in \mathbb{R}_+$, so that (A.2) still holds. Since $\text{supp}(F_{\eta,\ell}) = \text{supp}(F_\eta) \times \mathbb{R}_+$, we have $(u,l) \in \text{supp}(F_{\eta,\ell})$ and hence F_Y strictly increases at y. $\blacksquare$

We now prepare a series of lemmas for proving the two main results. The following is a restatement of Theorem 1.2 of Nagaev (1979):

Lemma A.2 *Let $X_1, \ldots, X_n$ be i.i.d. copies of a real-valued random variable X. Then for every $1 \leq q \leq 2$, $x > 0$, and $y > 0$, it holds that*

$$P\left(\sum_{i=1}^n X_i > x\right)$$
$$\leq nP(X > y)$$
$$+ \exp\left\{\frac{x}{y} - \left(\frac{x}{y} - \frac{n}{y}E\left[X1_{(|X|\leq y)}\right] + \frac{n}{y^q}E\left[|X|^q 1_{(|X|\leq y)}\right]\right)\log\left(1 + \frac{xy^{q-1}}{nE\left[|X|^q 1_{(|X|\leq y)}\right]}\right)\right\}.$$

Recall the function $r_1(s,t)$ defined in (4.2),

$$r_1(s,t) = E\left[\theta 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta\right)>\ell\right)}\right], \qquad (s,t) \in \mathbb{R}_+ \times \mathbb{R}.$$

Applying Lemma A.2, we obtain the following inequality:

Lemma A.3 *If $E\theta^q < \infty$ for some $1 < q \leq 2$, then for any $0 < \varepsilon < 1$, any $\lambda > 0$, all sufficiently large n, and uniformly for all $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$,*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta_i\right)>\ell_i\right)} - r_1(s,t)\right| > \varepsilon\right) \leq nP(\theta > \lambda n) + Cn^{-\frac{\varepsilon(q-1)}{2\lambda}},$$

where C is a positive constant irrespective of n.

Proof. It is important to note that every step in this proof holds uniformly for all $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$. For brevity, write

$$X = \theta 1_{\left(s\left(\rho t + \sqrt{1-\rho^2}\eta\right) > \ell\right)} \quad \text{and} \quad \tilde{X} = X - EX = X - r_1(s,t),$$

and introduce $\tilde{X}_1, \ldots, \tilde{X}_n$ to be i.i.d. copies of $\tilde{X}$. By Lemma A.2, we deal with the left-hand side, denoted by P_n, of the inequality under proof as

$$
\begin{aligned}
P_n &= P\left(\left|\sum_{i=1}^{n} \tilde{X}_i\right| > \varepsilon n\right) \\
&\leq nP\left(\tilde{X} > \lambda n\right) + nP\left(-\tilde{X} > \lambda n\right) \\
&\quad + 2\exp\left\{\frac{\varepsilon}{\lambda} - \left(\frac{\varepsilon}{\lambda} - \frac{1}{\lambda}\left|E\left[\tilde{X}1_{(|\tilde{X}|\leq \lambda n)}\right]\right|\right)\log\left(1 + \frac{\varepsilon(\lambda n)^{q-1}}{E\left[\left|\tilde{X}\right|^q 1_{(|\tilde{X}|\leq \lambda n)}\right]}\right)\right\}.
\end{aligned}
$$

Since $\tilde{X} \geq X - E\theta \geq -E\theta$, the term $nP(-\tilde{X} > \lambda n)$ vanishes for all large n. Since $E\tilde{X} = 0$, we have $E\left[\tilde{X}1_{(|\tilde{X}|\leq \lambda n)}\right] \to 0$. In addition, $E\left[\left|\tilde{X}\right|^q 1_{(|\tilde{X}|\leq \lambda n)}\right] \leq E(\theta + E\theta)^q < \infty$. It follows that, for sufficiently large n,

$$
\begin{aligned}
P_n &\leq nP(\theta > \lambda n) + 2\exp\left\{\frac{\varepsilon}{\lambda} - \frac{\varepsilon}{2\lambda}\log\left(1 + \frac{\varepsilon(\lambda n)^{q-1}}{E(\theta + E\theta)^q}\right)\right\} \\
&\leq nP(\theta > \lambda n) + Cn^{-\frac{\varepsilon(q-1)}{2\lambda}}.
\end{aligned}
$$

This ends the proof. ∎

Note that the function $r_1(s,t)$, which takes values in $[0, E\theta]$, is non-decreasing in $t \in \mathbb{R}$ and, under the last condition of Theorem 4.1, is strictly increasing in s over the range D_1 specified by $0 < r_1(s,t) < E\theta$. Thus, for $b \in (0, E\theta)$, the function $r_1(s,t)$ restricted to the range D_1 has a unique inverse with respect to s, denoted by $s_t(b)$. Furthermore, write

$$\tilde{r}_1(t) = r_1(\infty, t) = E\left[\theta 1_{\left(\rho t + \sqrt{1-\rho^2}\eta > 0\right)}\right]. \tag{A.3}$$

Clearly, for any $b \in (0, E\theta)$, both inverses $\tilde{r}_1^{\leftarrow}(b)$ and $\tilde{r}_1^{\rightarrow}(b)$ are finite. Since $\tilde{r}_1(t)$ is non-decreasing and left-continuous in t, it is easy to see the following:

- $\tilde{r}_1(\tilde{r}_1^{\leftarrow}(b)) \leq b$,
- $(\tilde{r}_1^{\leftarrow}(b), \infty) \subset \{t \in \mathbb{R} : \tilde{r}_1(t) \geq b\} \subset [\tilde{r}_1^{\leftarrow}(b), \infty)$,
- $\{t \in \mathbb{R} : \tilde{r}_1(t) > b\} = (\tilde{r}_1^{\rightarrow}(b), \infty)$.

Lemma A.4 *Assume $E\theta < \infty$ and the last condition of Theorem 4.1. Then for any fixed $b \in (0, E\theta)$ and any small $\delta > 0$, there exists some small $\varepsilon > 0$ such that, for all $t > \tilde{r}_1^{\leftarrow}(b)$,*

$$s_t(b) \geq \frac{\varepsilon}{t - \tilde{r}_1^{\leftarrow}(b - \delta)}.$$

Proof. For the given b and δ, choose some small $\varepsilon > 0$ such that $E\left[\theta 1_{(0<\ell\le\varepsilon)}\right] < \delta$. For any $t > \tilde{r}_1^{\leftarrow}(b)$, we derive

$$
\begin{aligned}
r_1\left(\frac{\varepsilon}{t-\tilde{r}_1^{\leftarrow}(b-\delta)}, t\right) &= E\left[\theta 1_{\left(\frac{\varepsilon\left(\rho t+\sqrt{1-\rho^2}\eta\right)}{\left(t-\tilde{r}_1^{\leftarrow}(b-\delta)\right)\ell}>1\right)}\left(1_{(\ell>\varepsilon)}+1_{(0<\ell\le\varepsilon)}\right)\right] \\
&< E\left[\theta 1_{\left(\frac{\rho t+\sqrt{1-\rho^2}\eta}{t-\tilde{r}_1^{\leftarrow}(b-\delta)}>1\right)}\right]+\delta \\
&\le E\left[\theta 1_{\left(\rho\tilde{r}_1^{\leftarrow}(b-\delta)+\sqrt{1-\rho^2}\eta>0\right)}\right]+\delta \\
&= \tilde{r}_1(\tilde{r}_1^{\leftarrow}(b-\delta))+\delta \\
&\le b.
\end{aligned}
$$

Thus, the desired inequality follows. ∎

Recall the function $r_2(u)$ defined in (4.7). For convenience, we introduce a modified version as follows: for any $u \in \mathbb{R}_+$ and any small $h \in \mathbb{R}$, say $|h| < 1$, define

$$
r_2(u;h) = E\left[\theta 1_{\left(\frac{\rho u}{\ell}+h>1\right)}\right]. \tag{A.4}
$$

It is easy to verify that $\lim_{h\downarrow 0} r_2(u;h) = r_2(u+)$ and $\lim_{h\uparrow 0} r_2(u;h) = r_2(u)$. Similarly to Lemma A.3, the following lemma considers the probability that the average

$$
\Sigma_n = \frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t+\sqrt{1-\rho^2}\frac{\eta_i}{f_n}\right)>1\right)}
$$

positively deviates from $r_2(st;\delta)$ or negatively deviates from $r_2(st;-\delta)$ for any small $\delta > 0$. In this lemma, by saying a property holds uniformly for $0 < s \ll f_n$ we mean that it holds uniformly for $0 < s \le \epsilon_n f_n$ for any given positive sequence $\epsilon_n = o(1)$.

Lemma A.5 *If $E\theta^q < \infty$ for some $1 < q \le 2$, then for any $0 < \delta < \varepsilon$, any $\lambda > 0$, all sufficiently large n, and uniformly for all $0 < s \ll f_n$ and $t \in \mathbb{R}$,*

$$
P\left(\Sigma_n - r_2(st;\delta) > \varepsilon\right) + P\left(\Sigma_n - r_2(st;-\delta) < -\varepsilon\right) \le nP(\theta > \lambda n) + Cn^{-\frac{(\varepsilon-\delta)(q-1)}{2\lambda}},
$$

where C is a positive constant irrespective of n.

Proof. It is important to note that every step in this proof holds uniformly for all $0 < s \ll f_n$ and $t \in \mathbb{R}$. For brevity, write

$$
X = \theta 1_{\left(\frac{s}{\ell}\left(\rho t+\sqrt{1-\rho^2}\frac{\eta}{f_n}\right)>1\right)} \qquad \text{and} \qquad \tilde{X} = X - EX,
$$

and introduce $\tilde{X}_1, \ldots, \tilde{X}_n$ to be i.i.d. copies of $\tilde{X}$. For the given $0 < \delta < \varepsilon$, choose some large M such that $E\left[\theta 1_{\left(\frac{|\eta|}{\ell} > M\right)}\right] \leq \delta$. Then it holds for all large n that

$$
\begin{aligned}
EX &\leq E\left[\theta 1_{\left(\rho\frac{st}{\ell} + \sqrt{1-\rho^2}\frac{s}{f_n}\frac{|\eta|}{\ell} > 1\right)} \left(1_{\left(\frac{|\eta|}{\ell} \leq M\right)} + 1_{\left(\frac{|\eta|}{\ell} > M\right)}\right)\right] \\
&\leq E\left[\theta 1_{\left(\rho\frac{st}{\ell} + \sqrt{1-\rho^2}\frac{s}{f_n}M > 1\right)}\right] + E\left[\theta 1_{\left(\frac{|\eta|}{\ell} > M\right)}\right] \\
&\leq r_2(st;\delta) + \delta
\end{aligned}
$$

and that

$$
\begin{aligned}
EX &\geq E\left[\theta 1_{\left(\rho\frac{st}{\ell} - \sqrt{1-\rho^2}\frac{s}{f_n}\frac{|\eta|}{\ell} > 1\right)} 1_{\left(\frac{|\eta|}{\ell} \leq M\right)}\right] \\
&\geq E\left[\theta 1_{\left(\rho\frac{st}{\ell} - \sqrt{1-\rho^2}\frac{s}{f_n}M > 1\right)}\right] - E\left[\theta 1_{\left(\frac{|\eta|}{\ell} > M\right)}\right] \\
&\geq r_2(st;-\delta) - \delta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&P\left(\Sigma_n - r_2(st;\delta) > \varepsilon\right) + P\left(\Sigma_n - r_2(st;-\delta) < -\varepsilon\right) \\
&\leq P\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i > \varepsilon - \delta\right) + P\left(\frac{1}{n}\sum_{i=1}^{n}\left(-\tilde{X}_i\right) > \varepsilon - \delta\right).
\end{aligned}
$$

We follow the proof of Lemma A.3 to derive similar upper bounds for the last two probabilities above. Finally, putting these two upper bounds together and keeping in mind that $-\tilde{X} \leq E\theta$, we conclude the proof. ∎

Proof of Theorem 4.1

We aim to establish the two-sided inequality (4.3). First we derive the corresponding upper bound for $P\left(L_n > nb\right)$. For any small $\delta > 0$, in terms of $r_1(s,t)$ defined in (4.2) and $\tilde{r}_1(t)$ defined in (A.3), we decompose it into three terms as

$$
\begin{aligned}
P\left(L_n > nb\right) &= P\left(L_n > nb, \xi \leq \tilde{r}_1^{\leftarrow}(b-\delta)\right) \\
&\quad + P\left(L_n > nb, r_1\left(\frac{S}{f_n}, \xi\right) < b - \delta, \xi > \tilde{r}_1^{\leftarrow}(b-\delta)\right) \\
&\quad + P\left(L_n > nb, r_1\left(\frac{S}{f_n}, \xi\right) \geq b - \delta, \xi > \tilde{r}_1^{\leftarrow}(b-\delta)\right) \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Note that, for all $s > 0$ and $t \leq \tilde{r}_1^{\leftarrow}(b-\delta)$,

$$
r_1(s,t) \leq \tilde{r}_1(\tilde{r}_1^{\leftarrow}(b-\delta)) \leq b - \delta. \tag{A.5}
$$

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Moreover, by the conditions on S and θ, we have $E\theta^q < \infty$ for $1 < q < 1 + \alpha$. Thus, by inequality (A.5) and Lemma A.3, it holds for some small $\lambda > 0$ and some large $C > 0$ that

$$
\begin{aligned}
I_1 &\leq \iint_{\mathbb{R}_+ \times (-\infty,\, \tilde{r}_1^{\leftarrow}(b-\delta)]} P\left(\frac{S}{f_n} \in ds\right) P\left(\xi \in dt\right) \\
&\quad \times P\left(\frac{1}{n}\sum_{i=1}^{n} \theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t + \sqrt{1-\rho^2}\eta_i\right) > 1\right)} - r_1(s,t) > \delta\right) \\
&\leq nP(\theta > \lambda n) + Cn^{-\frac{\delta(q-1)}{2\lambda}}.
\end{aligned}
$$

Similarly,

$$
I_2 \leq nP(\theta > \lambda n) + Cn^{-\frac{\delta(q-1)}{2\lambda}}.
$$

Choose some small $\lambda > 0$ such that $\frac{\delta(q-1)}{2\lambda} > \alpha$. Then by the conditions on S, θ and the condition $f_n = O(n)$, we obtain

$$
I_1 + I_2 = o\left(\overline{F_S}(n)\right) = o\left(\overline{F_S}(f_n)\right).
$$

It remains to prove that

$$
\limsup_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{I_3}{\overline{F_S}(f_n)} \leq \iint_{r_1(s,t) \geq b} \nu(ds) P\left(\xi \in dt\right). \tag{A.6}
$$

Clearly, for some large $M > 0$,

$$
\begin{aligned}
I_3 &\leq P\left(r_1\left(\frac{S}{f_n}, \xi\right) \geq b - \delta, \xi > \tilde{r}_1^{\leftarrow}(b-\delta)\right) \\
&\leq P\left(\xi > \frac{f_n}{M}\right) + \int_{\tilde{r}_1^{\leftarrow}(b-\delta)}^{\frac{f_n}{M}} P\left(r_1\left(\frac{S}{f_n}, t\right) \geq b - \delta\right) P\left(\xi \in dt\right) \\
&= I_{31} + I_{32}.
\end{aligned}
$$

By the conditions on S and ξ, we have

$$
I_{31} = o\left(\overline{F_S}(f_n)\right). \tag{A.7}
$$

To deal with I_{32}, for $t \in \mathbb{R}$ define $A_t = \{s \in \mathbb{R}_+ : r_1(s,t) \geq b - \delta\}$, which is a cross section of the set $A = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : r_1(s,t) \geq b - \delta\}$. By Lemma A.4, there is some small $\varepsilon_1 > 0$ such that, for all $\tilde{r}_1^{\leftarrow}(b-\delta) < t \leq \frac{f_n}{M}$ and all large n,

$$
f_n s_t(b-\delta) \geq \frac{\varepsilon_1 f_n}{\frac{f_n}{M} - \tilde{r}_1^{\leftarrow}(b-2\delta)} \sim \varepsilon_1 M,
$$

which can be sufficiently large by raising M. Thus, by Potter's bounds (see Proposition 2.2.3 of Bingham et al. (1987)), it holds for any small $\varepsilon_2 > 0$, all large M, all $\tilde{r}_1^{\leftarrow}(b-\delta) < t \leq \frac{f_n}{M}$, and all large n that

$$
\frac{P\left(r_1\left(\frac{S}{f_n}, t\right) \geq b - \delta\right)}{\overline{F_S}(f_n)} \leq \frac{P\left(\frac{S}{f_n} \geq s_t(b-\delta)\right)}{\overline{F_S}(f_n)}
$$

$$\leq (1+\varepsilon_2)\left((s_t(b-\delta))^{-\alpha-\varepsilon_2} \vee (s_t(b-\delta))^{-\alpha+\varepsilon_2}\right)$$
$$\leq C\left((t-\tilde{r}_1^{\leftarrow}(b-2\delta))^{\alpha+\varepsilon_2} \vee (t-\tilde{r}_1^{\leftarrow}(b-2\delta))^{\alpha-\varepsilon_2}\right),$$

with C a positive constant irrespective of t, where the last step is due to Lemma A.4. Applying the dominated convergence theorem, which is justified by the inequality above and the moment condition on ξ, and then applying relation (3.6), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{I_{32}}{\overline{F}_S(f_n)} &= \int_{\mathbb{R}} \lim_{n\to\infty} \frac{P\left(r_1\left(\frac{S}{f_n},t\right)\geq b-\delta\right)}{\overline{F}_S(f_n)} 1_{\left(\tilde{r}_1^{\leftarrow}(b-\delta)<t\leq \frac{f_n}{M}\right)} P\left(\xi\in dt\right) \\
&= \int_{\tilde{r}_1^{\leftarrow}(b-\delta)}^{\infty} \nu\left(A_t\right) P(\xi\in dt) \\
&\leq \iint_{r_1(s,t)\geq b-\delta} \nu(ds) P\left(\xi\in dt\right).
\end{aligned} \tag{A.8}$$

Putting (A.7)–(A.8) together gives (A.6).

Next we derive the corresponding lower bound for $P\left(L_n > nb\right)$. For any small $\varepsilon > 0$, define the set $\tilde{A} = \{(s,t)\in\mathbb{R}_+\times\mathbb{R}: r_1(s,t)>b+\varepsilon\}$. Then for $t\in\mathbb{R}$, write its cross section as $\tilde{A}_t = \{s\in\mathbb{R}_+: r_1(s,t)>b+\varepsilon\}$. We derive

$$\begin{aligned}
&P\left(L_n > nb\right) \\
&\geq P\left(L_n > nb, r_1\left(\frac{S}{f_n},\xi\right)>b+\varepsilon\right) \\
&= \iint_{\tilde{A}} P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t+\sqrt{1-\rho^2}\eta_i\right)>\ell_i\right)} > b\right) P\left(\frac{S}{f_n}\in ds\right) P\left(\xi\in dt\right).
\end{aligned}$$

Over $(s,t)\in\tilde{A}$, the i.i.d. summands $\theta_i 1_{\left(s\left(\rho t+\sqrt{1-\rho^2}\eta_i\right)>\ell_i\right)}$ in the tail probability above have a common mean $r_1(s,t)>b+\varepsilon$. Applying Lemma A.3, it holds uniformly for $(s,t)\in\tilde{A}$ that

$$\begin{aligned}
&P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t+\sqrt{1-\rho^2}\eta_i\right)>\ell_i\right)} > b\right) \\
&\geq P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(s\left(\rho t+\sqrt{1-\rho^2}\eta_i\right)>\ell_i\right)} - r_1(s,t) > -\varepsilon\right) \to 1.
\end{aligned}$$

It follows that

$$P\left(L_n > nb\right) \gtrsim \iint_{\tilde{A}} P\left(\frac{S}{f_n}\in ds\right) P\left(\xi\in dt\right) = \int_{\tilde{r}_1^{\rightarrow}(b+\varepsilon)}^{\infty} P\left(\frac{S}{f_n}\in\tilde{A}_t\right) P\left(\xi\in dt\right), \quad \text{(A.9)}$$

where the last step is due to the fact that $\tilde{r}_1(t)>b+\varepsilon$ if and only if $t>\tilde{r}_1^{\rightarrow}(b+\varepsilon)$. Moreover, from the definition of the function $r_1(s,t)$ in (4.2) it is easy to see that, for each fixed t, the

cross section $\tilde{A}_t$ as an interval is away from 0. Thus, by $\overline{F_S} \in \mathrm{RV}_{-\alpha}$,

$$\lim_{n\to\infty} \frac{P\left(\frac{S}{f_n} \in \tilde{A}_t\right)}{\overline{F_S}(f_n)} = \nu\left(\tilde{A}_t\right). \tag{A.10}$$

Applying Fatou's lemma to the right-hand side of (A.9) and applying (A.10), we obtain

$$\frac{P(L_n > nb)}{\overline{F_S}(f_n)} \gtrsim \int_{\tilde{r}_1^\to(b+\varepsilon)}^\infty \nu\left(\tilde{A}_t\right) P\left(\xi \in dt\right) = \iint_{r_1(s,t)>b+\varepsilon} \nu(ds) P\left(\xi \in dt\right).$$

Thus, the lower bound in (4.3) follows by letting $\varepsilon \downarrow 0$.

Proof of Theorem 4.2

We aim to establish the two-sided inequality (4.8). First we derive the corresponding upper bound for $P\left(L_n > nb\right)$. By the conditions on ξ and S, there is some auxiliary function $a(\cdot)$ such that the following limit relations, as $x \to \infty$, hold simultaneously:

- $0 < a(x) \uparrow \infty$,
- $\frac{x}{a(x)} \to \infty$,
- $P\left(S > \frac{x}{a(x)}\right) = o\left(\overline{F_\xi}(x)\right)$.

In terms of this auxiliary function $a(\cdot)$ and the function $r_2(u; h)$ introduced in (A.4), we split $P\left(L_n > nb\right)$ into three parts as

$$\begin{aligned}
P\left(L_n > nb\right) &= P\left(L_n > nb, S > \frac{f_n}{a(f_n)}\right) \\
&\quad + P\left(L_n > nb, r_2\left(\frac{S\xi}{f_n};\delta\right) < b - 2\delta, S \le \frac{f_n}{a(f_n)}\right) \\
&\quad + P\left(L_n > nb, r_2\left(\frac{S\xi}{f_n};\delta\right) \ge b - 2\delta, S \le \frac{f_n}{a(f_n)}\right) \\
&= J_1 + J_2 + J_3,
\end{aligned}$$

where $\delta > 0$ is arbitrarily fixed and small. Clearly,

$$J_1 \le P\left(S > \frac{f_n}{a(f_n)}\right) = o\left(\overline{F_\xi}(f_n)\right).$$

For J_2, by the conditions on ξ and θ, we have $E\theta^q < \infty$ for $1 < q < 1+\alpha$. As dealing with I_1 in the proof of Theorem 4.1, by Lemma A.5, it holds for any $\lambda > 0$ and some constant $C > 0$ that

$$J_2 \le \iint_{\left(0, \frac{f_n}{a(f_n)}\right] \times \mathbb{R}} P\left(S \in ds\right) P\left(\frac{\xi}{f_n} \in dt\right)$$

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$$\times P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t+\sqrt{1-\rho^2}\frac{\eta_i}{f_n}\right)>1\right)} - r_2(st;\delta) > 2\delta\right)$$

$$\leq nP(\theta > \lambda n) + Cn^{-\frac{\delta(q-1)}{2\lambda}}.$$

Choose some small $\lambda > 0$ such that $\frac{\delta(q-1)}{2\lambda} > \alpha$. Then it follows from the conditions on ξ, θ and the condition $f_n = O(n)$ that

$$J_2 = o\left(\overline{F_\xi}(n)\right) = o\left(\overline{F_\xi}(f_n)\right).$$

To deal with J_3, we derive

$$J_3 \leq \int_0^{\frac{f_n}{a(f_n)}} P\left(r_2\left(s\frac{\xi}{f_n};\delta\right) \geq b - 2\delta\right) P\left(S \in ds\right).$$

For $s \in \mathbb{R}_+$ define $A_s = \{t \in \mathbb{R} : r_2(st;\delta) \geq b - 2\delta\}$, which is a cross section of the set $A = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : r_2(st;\delta) \geq b - 2\delta\}$. Note that the inequality $r_2(u;\delta) \geq b - 2\delta$ implies that $u \geq u_0$ for some $u_0 > 0$. Then the event $\left(r_2\left(s\frac{\xi}{f_n};\delta\right) \geq b - 2\delta\right)$ appearing in J_3 implies that, for all $0 < s \leq \frac{f_n}{a(f_n)}$,

$$\xi \geq \frac{u_0}{s}f_n \geq u_0 a(f_n) \to \infty.$$

Then by Potter's bounds, it holds for any small $\varepsilon > 0$, all $0 < s \leq \frac{f_n}{a(f_n)}$, and all large n that

$$\frac{P\left(r_2\left(s\frac{\xi}{f_n};\delta\right) \geq b - 2\delta\right)}{\overline{F_\xi}(f_n)} \leq \frac{P\left(\xi \geq \frac{u_0}{s}f_n\right)}{\overline{F_\xi}(f_n)} \leq C\left(s^{\alpha+\varepsilon} \vee s^{\alpha-\varepsilon}\right)$$

for some positive constant C irrespective of s. Applying the dominated convergence theorem, which is justified by the inequality above and the moment condition on S, and then applying relation (3.6), we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{J_3}{\overline{F_\xi}(f_n)} &\leq \int_{\mathbb{R}_+} \lim_{n\to\infty} \frac{P\left(r_2\left(s\frac{\xi}{f_n};\delta\right) \geq b - 2\delta\right)}{\overline{F_\xi}(f_n)} 1_{\left(s\leq\frac{f_n}{a(f_n)}\right)} P\left(S \in ds\right) \\
&= \int_{\mathbb{R}_+} \nu(A_s) P\left(S \in ds\right) \\
&= \iint_{r_2(st;\delta)\geq b - 2\delta} P\left(S \in ds\right)\nu(dt).
\end{aligned}
$$

Putting these estimates together and letting $\delta \downarrow 0$, we obtain

$$\limsup_{n\to\infty} \frac{P(L_n > nb)}{\overline{F_\xi}(f_n)} \leq \iint_{r_2(st+)\geq b} P\left(S \in ds\right)\nu(dt),$$

which is the upper bound in (4.8).

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Next we derive the corresponding lower bound for $P(L_n > nb)$. For any small $\delta > 0$, define the set $\tilde{A} = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R} : r_2(st; -\delta) > b + 2\delta\}$. Then for $s \in \mathbb{R}_+$, write its cross section as $\tilde{A}_s = \{t \in \mathbb{R} : r_2(st; -\delta) > b + 2\delta\}$. We derive

$$
\begin{aligned}
&P(L_n > nb) \\
&\geq\; P\left(L_n > nb, r_2\left(\frac{S\xi}{f_n}; -\delta\right) > b + 2\delta, S \leq \frac{f_n}{a(f_n)}\right) \\
&=\; \int_0^{\frac{f_n}{a(f_n)}} \int_{\tilde{A}_s} P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t + \sqrt{1-\rho^2}\frac{\eta_i}{f_n}\right)>1\right)} > b\right) P\left(\frac{\xi}{f_n} \in dt\right) P(S \in ds).
\end{aligned}
$$

Similarly to the derivation for the lower bound in the proof of Theorem 4.1, by Lemma A.5, it holds uniformly for $0 < s \leq \frac{f_n}{a(f_n)}$ and $t \in \tilde{A}_s$ that

$$
\begin{aligned}
&P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t + \sqrt{1-\rho^2}\frac{\eta_i}{f_n}\right)>1\right)} > b\right) \\
&\geq\; 1 - P\left(\frac{1}{n}\sum_{i=1}^n \theta_i 1_{\left(\frac{s}{\ell_i}\left(\rho t + \sqrt{1-\rho^2}\frac{\eta_i}{f_n}\right)>1\right)} - r_2(st; -\delta) \leq -2\delta\right) \\
&\to\; 1.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
P(L_n > nb) \;&\gtrsim\; \int_0^{\frac{f_n}{a(f_n)}} \int_{\tilde{A}_s} P\left(\frac{\xi}{f_n} \in dt\right) P(S \in ds) \\
&=\; \int_0^{\frac{f_n}{a(f_n)}} P\left(\frac{\xi}{f_n} \in \tilde{A}_s\right) P(S \in ds). \tag{A.11}
\end{aligned}
$$

Moreover, from the definition of the function $r_2(u; h)$ in (A.4) it is easy to see that, for each fixed $s > 0$, the cross section $\tilde{A}_s$ as an interval is away from 0. Thus, by $\overline{F}_\xi \in \mathrm{RV}_{-\alpha}$,

$$
\lim_{n\to\infty} \frac{P\left(\frac{\xi}{f_n} \in \tilde{A}_s\right)}{\overline{F}_\xi(f_n)} = \nu\left(\tilde{A}_s\right). \tag{A.12}
$$

Applying Fatou's lemma to the right-hand side of (A.11) and applying (A.12), we obtain

$$
\frac{P(L_n > nb)}{\overline{F}_\xi(f_n)} \;\gtrsim\; \int_{\mathbb{R}_+} \nu\left(\tilde{A}_s\right) P(S \in ds) = \iint_{r_2(st;-\delta)>b+2\delta} P(S \in ds)\,\nu(dt).
$$

Thus, the lower bound in (4.8) follows by letting $\delta \downarrow 0$.

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