On the Class of Tail Dependence Matrices Generated by t and One-Factor Copula Families

N . D. Shyamalkumar[a], Siyang Tao[a]

[a]*Dept. of Statistics and Actuarial Science, The University of Iowa, USA*

Abstract

The tail dependence coefficient is a bivariate measure of dependence in the tails of two random variables, and the tail dependence matrix (TDM) is the array of such bivariate measures corresponding to a random vector. The TDM serves as a measure of multivariate tail dependence. Additionally, the space of TDMs corresponding to d-dimensional random vectors is a polytope with exponential in d number of facets and vertices. We show results that describe the subset of TDMs generated by some popular families of copulas; in some cases this subset is shown to be a surprisingly small part of the whole set of TDMs. For high dimensional cases, it can be proven that the subsets have no volume. This suggests another dimension along which to evaluate copula families for practical use.

Keywords: Tail dependence coefficient; Copula

1. Introduction

The study of the tail dependence structure has gained a lot of attention recently (see Embrechts et al. [1]). Among all the bivariate tail dependence measures, the tail dependence coefficient is one of the most popular choices in existing literature. For any random vector (X_1, X_2), with $X_1 \sim F_1$ and $X_2 \sim F_2$, the lower tail dependence coefficient, $\chi(X_1, X_2)$, is defined as

$$\chi(X_1, X_2) := \lim_{u \downarrow 0} \frac{\Pr\left(F_1(X_1) \le u, F_2(X_2) \le u\right)}{u},$$

given the limit exists. The tail dependence matrix (TDM), $(\chi(X_i, X_j))_{1 \le i \le j \le d}$, is the array of tail dependence coefficients corresponding to a random vector $\mathbf{X}_d = (X_1, \ldots, X_d)$. Hence, TDM serves as a d-dimensional measure of tail dependence. We denote the set of all d-dimensional TDMs by $\mathcal{T}_d$. For any $d \times d$ symmetric matrix $\mathbf{T}$, deciding whether $\mathbf{T} \in \mathcal{T}_d$ is referred to as the TDM realization problem and posted as an open question of significant interests in Embrechts et al. [1]. As observed in Strokorb [2] and Fiebig et al. [3], $\mathcal{T}_d$ is a $\binom{d}{2}$-dimensional polytope, and the explicit facet representations are given for $d \leq 6$. However, it is computationally hard to go further since the number of facets and vertices increasing dramatically; see Table 1 in Fiebig et al. [3].

In Fiebig et al. [3], it is shown that testing any matrix $\mathbf{T}_d$ with unit diagonal elements, which is a tail dependence matrix is equivalent to testing the matrix $\mathbf{T}_d/d$, which is a Bernoulli Compatible Matrix[1]. The membership problem of the Bernoulli Compatible Matrices is well-known as NP-Complete problem (see Pitowsky [4]), thus the TDM realization problem is believed to be an NP-hard problem since it is equivalent to proving a subset of the set of Bernoulli Compatible Matrices to be NP-complete. We denote the set of all d-dimensional Bernoulli Compatible Matrices to be $\mathcal{B}_d$.

In Lee [5], they first link the membership problem of $\mathcal{B}_d$ (or eventually as the TDM realization problem) to a Linear Programming (LP) formulation problem by using the vertex representation of $\mathcal{B}_d$. However, it is only solvable in reasonable time for $d \leq 20$ due to computational memory issues. As mentioned in Lee [5], various objective functions of the LP formulation can be chosen. In Krause et al. [6], they use the objective function as the max-norm distance from $\mathcal{B}_d$. Surprisingly, by applying the column generation method and some powerful prechecks, Krause et al. [6] shows that most instances of $d \leq 40$ could be done in 30 minutes of computing time. This doubled the ability for checking any arbitrary matrix. On the other hand, in Shyamalkumar and Tao [7], they choose the trivial objective function and give a comparison to the LP in Krause et al. [6]. They also show that if we restrict our attention to parametric classes with symmetric or sparsity properties, then the realization problem could be solved in polynomial time for any dimension. Interestingly, some parametrizations result in the constraint polytopes being independent of d.

The dependence structure of any random vector can be represented by a copula. The copula links the margins to the joint distribution; in other words, the copula is a multivariate distribution with all univariate margins being

[1]We note that the set of all Bernoulli Compatible Matrices is the convex hull of $\{\mathbf{x}\mathbf{x}^\mathsf{T} : \mathbf{x} \in \{0,1\}\}^d$.

standard uniform distribution. Let F be the multivariate cumulative distribution function (cdf) of a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with continuous margins F_i, for $i = 1, \ldots, d$. As described by Sklar [8], the copula C of F has the unique expression

$$C(u_1, \ldots, u_d) = F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right), \tag{1}$$

where $F_i^{-1}(\cdot)$, for $i = 1, \ldots, d$, are the inverse functions of the margins.

It has been established that the Gaussian copulas are tail independent (see McNeil et al. [9]). Thus, it is not theoretically interesting to study the TDMs generated by the Gaussian copulas. On the other hand, t copula is tail dependent with an explicit expression of the tail dependence coefficients given in McNeil et al. [9]. Hence, we are interested in studying the geometric structures of the TDMs generated by the t copula. Also, the factor models are widely used in many fields (see Krupskii and Joe [10]). We focus on the one-factor copula models in this paper.

In section 2, we study the geometric properties of the set of TDMs generated by the t copula. We first find the number of vertices that can be generated by the t copula for $d \leq 6$. In particular, we show that all of these feasible points are $\{0, 1\}$-valued and are subsets of the clique partition points. Importantly, we prove that it can be exploited to any dimension. In addition, we report the volume of the feasible points for $d \leq 6$. In section 3, we consider the one-factor copula families. We focus on the linking copula being the Family BB1 which was presented in Joe and Hu [11]. We find the set of all feasible $\{0, 1\}$-valued vertices for any dimension as a subset of the clique partition points. Then we calculate the exact volume for $d = 3$ and show the feasible region for $d = 4$. By choosing different linking copulas, the TDMs generated by the one-factor copulas yield significantly different volumes on $d = 3$.

Notation: All vectors and matrices are boldfaced. We denote by $\mathbf{I}$ the identity matrix. We denote $\mathbf{1}$ and $\mathbf{J}$ be the vectors and matrices with all elements equal to 1, respectively. Also, by $\mathbf{0}$ we denote vectors and matrices with all of the elements equal to 0. We denote the floor function by $\lfloor \cdot \rfloor$, the ceiling function by $\lceil \cdot \rceil$, and the indicator function of a set A by I_A.

2. Richness of the *t* Copula

In this section, we study the TDMs generated by the t copula. Our goal is to find the number of vertices and the approximate percentage volume of $\mathcal{T}_d$, for $3 \leq d \leq 6$, that can be generated from the t copula. Interestingly, we were able to generalize geometric properties for the vertices (provided in Theorem 1 below) which can be generated from the t copula for any dimension. The d-dimensional t copula is defined as

$$C^t_{\nu,P}(u_1,\dots,u_d) = \mathbf{t}_{\nu,\mathbf{P}}\left(t_\nu^{-1}(u_1),\dots,t_\nu^{-1}(u_d)\right),$$

where $\mathbf{t}_{\nu,\mathbf{P}}$ is the joint cdf of the random vector $\mathbf{X} \sim \mathbf{t}_d(\nu,\mathbf{0},\mathbf{P})$ with the correlation matrix $\mathbf{P} = (P_{ij})$, and t_ν is the cdf of a standard univariate t distribution with ν degrees of freedom. We note that in McNeil et al. [9], the tail dependence coefficients $(\chi(X_i,X_j))_{1\leq i,j\leq d}$ corresponding to $\mathbf{X}$ are shown to be

$$\chi(X_i,X_j) = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-P_{ij})}{1+P_{ij}}}\right), \qquad 1 \leq i,j \leq d. \tag{2}$$

Remark 1. The correlation matrix $\mathbf{P}$ is positive semi-definite with $P_{ij} \in [-1,1]$, for $1 \leq i,j \leq d$, and the tail dependence coefficients associated to $\mathbf{P}$, $\chi(X_i,X_j) := \phi(\nu,P_{ij})$, for $1 \leq i,j \leq d$, are monotone with respect to the 2 arguments; that is, for fixed ν, $\phi(\nu,\cdot)$ increases as P_{ij} increases, and for fixed P_{ij}, $\phi(\cdot,P_{ij})$ decreases as ν increases. Also, some extreme values are $\phi(\cdot,-1) = 0$, $\phi(\cdot,1) = 1$ and for $\rho < 1$, $\phi(\infty,\rho) = 0$. Note that even for negative correlation, there is tail dependence of the t copula.

Remark 2. By using the independence of the d-dimensional random vector $\mathbf{X}$, the origin of $\mathbb{R}^{\binom{d}{2}}$ is always a vertex of $\mathcal{T}_d$. However, the origin can not be generated by the t copula. In fact, for any $\nu \in \mathbb{R}^+$, if we consider $\mathbf{P}$ to be the equi-correlation matrix with off-diagonal elements ρ, then the random vector $\mathbf{X}$ follows $\mathbf{t}_d(\nu,\mathbf{0},(1-\rho)\mathbf{I}_d+\rho\mathbf{J})$. Thus the TDM corresponding to $\mathbf{X}$ has the same off-diagonal elements, say χ. Since the variance of the sum of elements in $\mathbf{X}$ is non-negative, we could derive the lower bound for ρ which is given by $-1/(d-1)$. The monotonicity with respect to P_{ij} in Remark 1 implies

$$\chi \geq 2t_{\nu+1}\left(-\sqrt{\frac{d(\nu+1)}{d-2}}\right) > 0.$$

Hence, the origin can not be generated by the t copula.

In Fiebig et al. [3], using the connection between $\mathcal{T}_d$ and CUT polytopes (see Deza and Laurent [12]) and correlation polytopes, they derive the vertices and facets of the TDM polytopes for dimensions up to six. The limit of six is due to the existence of descriptions of the CUT polytopes until dimension seven (see Deza and Laurent [12]). It is known that the numbers of vertices and facets grow exponentially with increasing dimension.

Accordingly, we check the feasibility for all vertices of $\mathcal{T}_d$, for $d \leq 6$, in the following steps. For any vertex of $\mathcal{T}_d$, for $d \leq 6$, we get the corresponding matrix $\mathbf{P}$ by the inverse function of (2), and we check the positive semi-definiteness of $\mathbf{P}$. In Table 1, we report the number of vertices which could be generated by the t copula. While dramatically increasing the number of vertices of the set of all TDMs, the number of feasible vertices generated by the t copula only increases by double.

Table 1: Number and percentage of feasible vertices generated by the t copula

d	3	4	5	6
# of vertices	5	15	214	28895
# of feasible vertices	4	8	16	32
percentage	80%	53%	7.5%	0.11%

It is interesting to point out that all the feasible vertices for $d \leq 6$ have some geometric properties related to the clique partition points. A set $\{A_1, \ldots, A_k\}$, for $k \geq 1$, is a partition of a set A if $A_i \cap A_j = \emptyset$, for $i \neq j$ and $\cup_{i=1}^{k} A_i = A$; while $k = 2$, $\{A_1, A_2\}$ is said to be a bipartition of A. For any partition $\{A_1, \ldots, A_k\}$, for $k \geq 1$, of the set $[d] := \{1, \ldots .d\}$, the clique partition point $\pi(\{A_1, \ldots, A_k\}) \in \{0, 1\}^{\binom{d}{2}}$ is defined by

$$\pi(\{A_1, \ldots, A_k\})_{ij} := \sum_{r=1}^{k} I_{\{i,j\} \subset A_r}, \quad 1 \leq i < j \leq n,$$

see Fiebig et al. [3]. We denote the set of all such points in $\{0, 1\}^{\binom{d}{2}}$ by

$$\mathcal{C}_d := \{\pi(\{A_1, \ldots, A_k\}) : \{A_1, \ldots, A_k\} \text{ partition of } [d]\};$$

also, for all such points in $\{0, 1\}^{\binom{d}{2}}$ for the bipartition, we define

$$\mathcal{C}_d^b := \{\pi(\{A_1, A_2\}) : \{A_1, A_2\} \text{ bipartition of } [d]\}. \tag{3}$$

We observed that the set of the feasible vertices with respect to the t copula is $\mathcal{C}_d^b$, for $d \leq 6$. It is easy to determine that the number of the possible bipartitions of $[d]$ is 2^{d-1}, thus the number of the feasible vertices is also $|\mathcal{C}_d^b| = 2^{d-1}$ as shown in Table 1. Surprisingly, the vertices of the form (3) could be applied to any dimension, which covers all $\{0, 1\}$-valued vertices.

Lemma 1. *All points of the set $\mathcal{C}_d^b$ are feasible vertices with respect to the t copula.*

Proof. Let $\nu > 0$ and $\{A_1, A_2\}$ be a bipartition of $[d]$. We need to prove that $\pi(\{A_1, A_2\})$ is a feasible vertex with respect to the t copula. It is trivial when $A_1 \in \{\emptyset, [d]\}$ because the vertex $\mathbf{1}_{1\times\binom{d}{2}}$ can be generated by the t copula. For $A_1 \notin \{\emptyset, [d]\}$, without loss of generality, we could assume $A_1 = [d']$, for $0 < d' < d$, then $A_2 = \{d'+1, \ldots, d\}$. Now we set $\mathbf{P}_d = (P_{ij})$, where

$$P_{ij} = \begin{cases} 1 & 1 \le i, j \le d' \text{ or } d'+1 \le i, j \le d; \\ -1 & \text{otherwise.} \end{cases}$$

Then by the TDMs given in (2), we can easily verify that the tail dependence coefficients corresponding to the t copula with the matrix $\mathbf{P}$ is $\pi(\{A_1, A_2\})$. All that remains to be shown is the matrix $\mathbf{P}_d = \mathbf{x}\mathbf{x}^\top$ where $\mathbf{x} = (\mathbf{1}_{d'}^\top, -\mathbf{1}_{d-d'}^\top)^\top$ is positive semi-definite, which is a well-known result of linear algebra. □

Theorem 1. *Let $v \in \{0,1\}^{\binom{d}{2}}$ be a vertex of $\mathcal{T}_d$. Then v can be generated by the t copula if and only if v is in $\mathcal{C}_d^b$.*

Proof. The if part follows directly from Lemma 1. For the only if part, we note that $\mathcal{T}_d \cap \{0,1\}^{\binom{d}{2}} = \mathcal{C}_d$ (see Proposition 21 of Fiebig et al. [3]). Thus, we need to prove $v \notin \mathcal{C}_d \backslash \mathcal{C}_d^b$ which is equivalent to showing that the matrix

$$\mathbf{J} := \begin{bmatrix} \mathbf{J}_{d_1} & -\mathbf{J}_{d_1,d_2} & \cdots & -\mathbf{J}_{d_1,d_{k-1}} & -\mathbf{J}_{d_1,d_k} \\ -\mathbf{J}_{d_2,d_1} & \mathbf{J}_{d_2} & \cdots & -\mathbf{J}_{d_2,d_{k-1}} & -\mathbf{J}_{d_2,d_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathbf{J}_{d_{k-1},d_1} & -\mathbf{J}_{d_{k-1},d_2} & \cdots & \mathbf{J}_{d_{k-1}} & -\mathbf{J}_{d_{k-1},d_k} \\ -\mathbf{J}_{d_k,d_1} & -\mathbf{J}_{d_k,d_2} & \cdots & -\mathbf{J}_{d_k,d_{k-1}} & \mathbf{J}_{d_k} \end{bmatrix},$$

for $k \ge 3$ and $\sum_{i=1}^k d_i = d$ is not positive semi-definite. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d)^\top$, where

$$\mathbf{x}_j := (1/d_j)\mathbf{1}_{d_j}^\top, \quad \text{for } j = 1, \ldots, k.$$

Then $\mathbf{x}^\top \mathbf{J} \mathbf{x} = k(2-k) < 0$ completes the proof. □

For dimensions up to six, the full description of the TDM polytope allows for an ideal study of approximate volume as described in the following. We randomly generate 100,000 points from the polytope $\mathcal{T}_d$, for $d \leq 6$, using the function `cprnd` (see Benham [13] of MATLAB®). In table 2, we report the percentage of the feasible points with respect to the t copula. As d increases, the percentage decreases for any ν; with no feasible points for $\nu = \infty$.

Table 2: Percentage of feasible points generated from $\mathcal{T}_d$, for $d \leq 6$, with respect to the t copula

$\nu \backslash d$	3	4	5	6
2	77.08%	33.98%	7.81%	0.91%
5	73.24%	28.51%	5.10%	0.46%
10	72.13%	26.69%	4.23%	0.31%
100	71.58%	24.80%	3.94%	0.26%
∞	0	0	0	0

3. Richness of the One-factor Copula Models

In this section, we show the geometric properties of the set of TDMs generated by the one-factor copula models. We first note that the equation (1) can also be written as

$$C(u_1, \ldots, u_d) = \Pr\left(U_1 \leq u_1, \ldots, U_d \leq u_d\right),$$

where U_i, for $i = 1, \ldots, d$, are standard uniform distributions. Henceforth, we consider the random vector $\mathbf{U} := (U_1, \ldots, U_d)$ instead of $\mathbf{X}$. In the one-factor copula models, the random variables U_i, for $i = 1, \ldots, d$, are assumed to be conditionally independent given a latent random variable $V \sim U(0, 1)$. Hence the one-factor copula can be expressed as

$$C(u_1, \ldots, u_d) = \int_0^1 \prod_{i=1}^{d} C_{i|0}(u_i|v)dv,$$

where $C_{i0}(u_i, v)$, for $i = 1, \ldots, d$, are bivariate copulas and $C_{i|0}(u_i|v) := \partial C_{i0}(u_i, v)/\partial v$, for $i = 1, \ldots, d$, are the conditional distributions. Since $C_{i0}(u_i, v)$, for $i = 1, \ldots, d$, are the joint cdfs linking the random variables U_i and V, they are called the *linking copulas* of the one-factor models.

We consider the linking copulas to be the Family BB1 of the bivariate two-parameter families which presented in section 5.2 of Joe [14]. We note that the Family BB1 is of the form

$$C_{i0}(u,v;\theta_i,\delta_i) = \left\{1+\left[\left(u^{-\theta_i}-1\right)^{\delta_i}+\left(v^{-\theta_i}-1\right)^{\delta_i}\right]^{1/\delta_i}\right\}^{-1/\theta_i}, \quad \theta_i>0,\ \delta_i\geq 1,\ i=1,\dots,d. \tag{4}$$

Remark 3. If we consider the Family BB1 rather than using it as the linking copulas, then it is easy to find its tail dependence coefficient. In Joe and Hu [11], they show that the copula of the form

$$C(u,v) = \eta\left(\eta^{-1}(u)+\eta^{-1}(v)\right) \tag{5}$$

has the tail dependence coefficient which is given by $2\lim_{s\uparrow\infty}[\eta'(2s)/\eta'(s)]$. Since the expression in (4) can be written as the form (5) with $\eta(s) = \left(1+s^{1/\delta}\right)^{-1/\theta}$ and $\eta^{-1}(s) = \left(s^{-\theta}-1\right)^{\delta}$, then the tail dependence coefficients of the Family BB1 are

$$2\lim_{s\uparrow\infty}\frac{\eta'(2s)}{\eta'(s)} = 2\lim_{s\uparrow\infty}\left(\frac{1+(2s)^{1/\delta}}{1+s^{1/\delta}}\right)^{-1/\theta-1}2^{1/\delta-1} = 2^{-1/(\theta\delta)}.$$

Note that we drop the subscripts to simplify the notation.

Since the tail dependence coefficients χ_{ij}, for $1\leq i<j\leq d$, of the one-factor copulas are

$$\chi_{ij} = \lim_{u\downarrow 0}\frac{1}{u}\int_0^1 C_{i|0}(u|v)C_{j|0}(u|v)dv = \lim_{u\downarrow 0}\int_0^{1/u} C_{i|0}(u|uw)C_{j|0}(u|uw)dw, \tag{6}$$

then we will show the expressions and the properties of the conditional distributions $C_{i|0}(u|uw)$, for $i=1,\dots,d$, in the following 3 lemmas.

Lemma 2. *The conditional distributions $C_{i|0}(u|uw)$, for $i=1,\dots,d$, are given by*

$$C_{i|0}(u|uw) = \left\{1+\left[\left(u^{-\theta_i}-1\right)^{\delta_i}+\left((uw)^{-\theta_i}-1\right)^{\delta_i}\right]^{1/\delta_i}\right\}^{-1/\theta_i-1} \left[\left(u^{-\theta_i}-1\right)^{\delta_i}+\left((uw)^{-\theta_i}-1\right)^{\delta_i}\right]^{1/\delta_i-1}\left((uw)^{-\theta_i}-1\right)^{\delta_i-1}(uw)^{-\theta_i-1}.$$

Proof. We first take the derivative with respect to v of $C_{i0}(u,v;\theta_i,\delta_i)$. Then the result follows by replacing v by uw. □

Lemma 3. *$|C_{i|0}(u|uw)|$, for $i = 1, \ldots, d$, are bounded by the functions $g_i(\cdot)$, for $i = 1, \ldots, d$, respectively where*

$$g_i(w) := \begin{cases} 1, & 0 \le w \le 1; \\ w^{-1-\theta_i\delta_i}, & w \ge 1. \end{cases}$$

Proof. Without loss of generality, we are going to prove $C_{1|0}(u|uw)$ is bounded by $g_1(\cdot)$. We drop the subscripts of θ_1 and δ_1 to simplify the notation. Let $s = \eta^{-1}(uw) = \left((uw)^{-\theta} - 1\right)^{\delta}$, then $u = \eta(s)/w = \left(1 + s^{1/\delta}\right)^{-1/\theta}/w$ which can also be written as $u^{-\theta} = (1+s^{1/\delta})w^{\theta}$, and it is clear that when u decreases to 0, s increases to infinity. Hence

$$C_{1|0}(u|uw) = \left(\frac{1 + \left[\left(\left(1 + s^{\frac{1}{\delta}}\right) w^{\theta} - 1\right)^{\delta} + s\right]^{\frac{1}{\delta}}}{1 + s^{\frac{1}{\delta}}}\right)^{-\frac{1}{\theta}-1} \left(\frac{\left(\left(1 + s^{\frac{1}{\delta}}\right) w^{\theta} - 1\right)^{\delta} + s}{s}\right)^{\frac{1}{\delta}-1}, \quad (7)$$

therefore $|C_{1|0}(u|uw)| \le \left(2^{-1}s^{-1/\delta} + 2^{-1}\left(1+m\right)^{1/\delta}\right)^{-1/\theta-1} \left(1+m\right)^{1/\delta-1}$, where $m = \left(\left(1 + s^{1/\delta}\right) w^{\theta} - 1\right)^{\delta}/s$. Note that $s^{-1/\delta} \in (0, 1]$. Since

$$1 + m = 1 + \left(\left(1 + s^{-1/\delta}\right) w^{\theta} - s^{-1/\delta}\right)^{\delta} \ge 1 + w^{\theta\delta} \ge \begin{cases} 1, & 0 \le w \le 1; \\ w^{\theta\delta}, & w \ge 1, \end{cases}$$

we have

$$\begin{aligned} |C_{1|0}(u|uw)| &\le \left(2^{-1}s^{-1/\delta} + 2^{-1}\left(1 + w^{\theta\delta}\right)^{1/\delta}\right)^{-1/\theta-1} \left(1 + w^{\theta\delta}\right)^{1/\delta-1} \\ &\le \begin{cases} \left(2^{-1}s^{-1/\delta} + 2^{-1}\right)^{-1/\theta-1}, & 0 \le w \le 1; \\ \left(2^{-1}s^{-1/\delta} + 2^{-1}w^{\theta}\right)^{-1/\theta-1} w^{\theta-\theta\delta}, & w \ge 1, \end{cases} \\ &\le \begin{cases} 1, & 0 \le w \le 1; \\ w^{-1-\theta\delta}, & w \ge 1. \end{cases} \end{aligned}$$

□

Lemma 4. *The limits of the conditional distributions $C_{i|0}(u|uw)$, for $i = 1, \ldots, d$, when u goes down to 0 are given by*

$$\lim_{u \downarrow 0} C_{i|0}(u|uw) = \left(1 + w^{\theta_i\delta_i}\right)^{-\frac{1}{\theta_i\delta_i}-1}, \ \textit{for } i = 1, \ldots, d.$$

Proof. For any $i = 1, \dots, d$, it follows from (7) that

$$\lim_{s\uparrow\infty} C_{i|0}(u|uw) = \left(1 + w^{\theta_i\delta_i}\right)^{\frac{1}{\delta_i}\left(-\frac{1}{\theta_i}-1\right)+\frac{1}{\delta_i}-1} = \left(1 + w^{\theta_i\delta_i}\right)^{-\frac{1}{\theta_i\delta_i}-1}.$$

□

Remark 4. In general, for all linking copula of the form

$$C_{i0}(u, v) = \eta_i\left(\eta_i^{-1}(u) + \eta_i^{-1}(v)\right), \quad \text{for } i = 1, \dots, d, \tag{8}$$

the limits of the conditional distributions $C_{i|0}(u|uw)$, for $i = 1, \dots, d$, are given by

$$\lim_{u\downarrow 0} C_{i|0}(u|uw) = \lim_{u\downarrow 0} \frac{\eta_i'(\eta_i^{-1}(u) + \eta_i^{-1}(uw))}{\eta_i'(\eta_i^{-1}(uw))}, \quad \text{for } i = 1, \dots, d.$$

Remark 5. Another method for calculating the limit of $C_{i|0}(u|uw)$, for $i = 1, \dots, d$, is followed by using section 8.3.2 of Joe [15]. The tail dependence functions of the linking copulas $C_{i0}(u_1, u_2)$, for $i = 1, \dots, d$, are given by

$$b_i(w_1, w_0) := \lim_{u\downarrow 0} C_{i0}(uw_1, uw_0)/u = \left(w_1^{-\theta_i\delta_i} + w_0^{-\theta_i\delta_i}\right)^{-1/(\theta_i\delta_i)},\ i = 1, \dots, d,$$

then as $u \downarrow 0$, the conditional distributions

$$C_{i|0}(uw_1|uw_0) \sim \frac{\partial b(w_1, w_0)}{\partial w_0} = \left(1 + (w_0/w_1)^{-\theta_i\delta_i}\right)^{-\frac{1}{\theta_i\delta_i}-1},\ i = 1, \dots, d.$$

Hence, the limits of the conditional distributions would be

$$\lim_{u\downarrow 0} C_{i|0}(u|uw) = \left(1 + w^{\theta_i\delta_i}\right)^{-\frac{1}{\theta_i\delta_i}-1}\ \ i = 1, \dots, d,$$

which yield the same result as Lemma 4.

Theorem 2. *The tail dependence coefficients χ_{ij}, for $1 \le i < j \le d$, of the one-factor copula with the linking copula being the Family BB1 are given by*

$$\chi_{ij} = \int_0^\infty \left(1 + w^{\theta_i\delta_i}\right)^{-\frac{1}{\theta_i\delta_i}-1}\left(1 + w^{\theta_j\delta_j}\right)^{-\frac{1}{\theta_j\delta_j}-1} dw.$$

Proof. We note that from the Lemma 3

$$\int_0^\infty g_i(w)g_j(w)dw \leq 1 + \frac{1}{2\theta\delta + 1} < \infty,$$

where $\theta\delta := \min\{\theta_i\delta_i, \theta_j\delta_j\}$, for $1 \leq i < j \leq d$. Thus the tail dependence coefficients χ_{ij}, for $1 \leq i < j \leq d$, are followed by (6), Lemma 4 and the dominated convergence theorem. □

Theorem 2 indicates the formula of the tail dependence coefficient χ_{ij}, for $1 \leq i < j \leq d$, depending on θ_i, δ_i, θ_j and δ_j, for $1 \leq i < j \leq d$. But the pairs of parameters $\{\theta_i, \delta_i\}$, for $i = 1, \ldots, d$, come from the same linking copula. Hence, we should re-parameterize the expression which is shown in the following corollary. We define

$$\Psi(x,y) := \int_0^\infty (1 + w^x)^{-x^{-1}-1} (1 + w^y)^{-y^{-1}-1} dw, \quad x, y > 0.$$

Corollary 1. *The tail dependence coefficients χ_{ij}, for $1 \leq i < j \leq d$, of the one-factor copula with the linking copula being the Family BB1 are given by $\chi_{ij} = \Psi(a_i, a_j)$, where $a_i = \theta_i\delta_i$ and $a_j = \theta_j\delta_j$, for $1 \leq i < j \leq d$.*

In order to study the geometric properties of the set of TDMs generated by the one-factor copula family, we show necessary properties for $\Psi(\cdot, \cdot)$ and its related functions. We denote the function $\psi(\cdot, \cdot, \cdot)$ by

$$\psi(x, y, w) := (1 + w^x)^{-x^{-1}-1} (1 + w^y)^{-y^{-1}-1}, \qquad x, y, w > 0.$$

Note that $\Psi(x, y) = \int_0^\infty \psi(x, y, w)dw$, for $x, y > 0$. The partial derivative with respect to x is given by

$$\frac{\partial\psi(x, y, w)}{\partial x} = \frac{(1 + w^x)^{-x^{-1}-1}}{(1 + w^y)^{y^{-1}+1}} \left(x^{-2} \ln (1 + w^x) - \left(x^{-1} + 1\right) \frac{w^x \ln w}{1 + w^x} \right). \quad (9)$$

Lemma 5. *For any fixed y, $\Psi(x, y)$ is strictly increasing continuously with respect to x on $(0, \infty)$.*

Proof. The continuity is trivial. We need to prove that $\partial\Psi(x, y)/\partial x > 0$. The continuity of $\psi(x, y, w)$ and its partial derivatives with respect to x and w

allows us to apply apply the Leibniz integral rule. Therefore, it is equivalent to prove that $\int_0^\infty (\partial\psi(x,y,w)/\partial x)\, dw > 0$. Then, from (9), we need to show

$$\int_0^\infty \frac{(1+w^x)^{-x^{-1}-1}}{(1+w^y)^{y^{-1}+1}} \left(x^{-2}\ln(1+w^x) - \left(x^{-1}+1\right)\frac{w^x \ln w}{1+w^x} \right) dw > 0.$$

Let $w^x = z$, then $dw = x^{-1}z^{1/x-1}dz$. Hence the left hand side of the last inequality is equal to

$$\begin{aligned}
&\int_0^\infty \left(1+z^{y/x}\right)^{-1/y-1} x^{-3}(1+z)^{-1/x-1}\left(\ln(1+z)\, z^{1/x-1} - (x+1)\frac{z^{1/x}\ln z}{1+z}\right) dz \\
=&\int_0^1 \left(1+z^{y/x}\right)^{-1/y-1} x^{-3}(1+z)^{-1/x-1}\left(\ln(1+z)\, z^{1/x-1} - (x+1)\frac{z^{1/x}\ln z}{1+z}\right) dz \\
&+\int_1^\infty \left(1+z^{y/x}\right)^{-1/y-1} x^{-3}(1+z)^{-1/x-1}\left(\ln(1+z)\, z^{1/x-1} - (x+1)\frac{z^{1/x}\ln z}{1+z}\right) dz
\end{aligned} \tag{10}$$

Let $u = 1/z$, then $dz = -u^{-2}du$. Thus the second term of (10) is given by

$$\int_0^1 \left(1+u^{y/x}\right)^{-1/y-1} x^{-3}(1+u)^{-1/x-1}\left(\ln(1+u) - \ln u + (x+1)\frac{\ln u}{1+u}\right) u^{1/x+y/x}du;$$

whence (10) becomes

$$\begin{aligned}
&\int_0^1 \left(1+z^{y/x}\right)^{-1/y-1} x^{-3}(1+z)^{-1/x-1} z^{1/x} \\
&\cdot\left(\left(z^{y/x}+1/z\right)\ln(1+z) - z^{y/x}\ln z - (x+1)\frac{1-z^{y/x}}{1+z}\ln z\right) dz.
\end{aligned}$$

Hence the result is followed by $\left(z^{y/x}+1/z\right)\ln(1+z) - z^{y/x}\ln z - (x+1)\frac{1-z^{y/x}}{1+z}\ln z > 0$, for $z \in [0,1]$ and $x, y > 0$. □

Lemma 6. *For any fixed y, $\Psi_1(x,y) := \int_0^1 \psi(x,y,w)dw$ is strictly increasing with respect to x on $(0,\infty)$.*

Proof. By the same argument as Lemma 5, we need to show

$$\int_0^1 \frac{(1+w^x)^{-x^{-1}-1}}{(1+w^y)^{y^{-1}+1}} \left(x^{-2}\ln(1+w^x) - \left(x^{-1}+1\right)\frac{w^x \ln w}{1+w^x} \right) dw > 0.$$

Again, let $w^x = z$, then it is equivalent to prove

$$\int_0^1 \frac{x^{-3}\,(1+z)^{-1/x-1}\,z^{1/x}}{\left(1+z^{y/x}\right)^{1/y+1}} \left(\ln(1+z)\,z^{-1} - (x+1)\,\frac{\ln z}{1+z}\right) dz > 0.$$

The monotone increasing follows by $\ln(1+z)\,z^{-1} - (x+1)\,\frac{\ln z}{1+z} > 0$, for $z \in [0,1]$ and $x, y > 0$. $\square$

Remark 6. By symmetry, for any fixed x, $\Psi(x,y)$ and $\Psi_1(x,y)$ are also strictly increasing continuously with respect to y on $(0,\infty)$. Also, for any $x, y \in \mathbb{R}_+$, $\Psi(x,y)$ and $\Psi_1(x,y)$ are both bounded from below by $\Psi(0,0) = \Psi_1(0,0) = 0$ and bounded from above by $\Psi(\infty,\infty) = \Psi_1(\infty,\infty) = 1$. That means $\Psi(x,y), \Psi_1(x,y) \in (0,1)$, for any $x, y \in (0,\infty)$.

Lemma 7. $\Psi(x,\infty) = \Psi_1(x,\infty)$ *is strictly increasing on* $(0,\infty)$.

Proof. We note that $\Psi(x,\infty) = \Psi_1(x,\infty) = \int_0^1 (1+w^x)^{-1/x-1} dw$. By a similar argument as Lemma 5, we need to prove

$$\int_0^1 (1+w^x)^{-1/x-1} \left(x^{-2}\ln(1+w^x) - (1/x+1)\,\frac{w^x \ln w}{1+w^x}\right) dw > 0.$$

By the argument of Lemma 6, this is true. $\square$

The following Proposition shows that the above properties can lead us to check the extreme values of $\Psi(\cdot,\cdot)$. We state this without proof since it directly follows from Remark 6 and Lemma 7.

Proposition 1. $\Psi(x,y) = 0$ if and only if $x = 0$ or $y = 0$. In addition, $\Psi(x,y) = 1$ if and only if $x = y = \infty$.

Remark 7. From Proposition 1, one could easily check the feasibility of all $\{0,1\}$-valued vertices for the set of TDMs. For instance, on dimension 4, the vertex $(\chi_{12}, \chi_{13}, \chi_{14}, \chi_{23}, \chi_{24}, \chi_{34}) = (1,0,0,0,0,1)$ could not be generated from the one-factor family. The proof is simple, if $\lambda_i = \theta_i \delta_i$, for $i = 1,2,3,4$, are the 4 parameters of the one factor family, then $\chi_{12} = \chi_{34} = 1$ indicates that $\lambda_i = \infty$, for $i = 1,2,3,4$, which contradicts with $\chi_{13} = \chi_{14} = \chi_{23} = \chi_{24} = 0$.

As shown in Fiebig et al. [3], the intersection between $\mathcal{T}_d$ and all $\{0,1\}$-valued points is the set of the d-dimensional clique partition points. Hence it is worth pointing out all the feasible vertices that can be generated from the one-factor copula as a subset of $\mathcal{C}_d$. We define

$$\mathcal{C}_d^o := \left\{ \pi(\{A_1, \ldots, A_k\}) : \{A_1, \ldots, A_k\} \text{ partition of } [d],\ \sum_{i=1}^{k} I_{|A_i|>1} \leq 1 \right\}.$$

Theorem 3. *Let $v \in \{0,1\}^{\binom{d}{2}}$ be a vertex of $\mathcal{T}_d$. Then v can be generated by the one-factor copula with the linking copula being the Family BB1 if and only if v is in $\mathcal{C}_d^o$.*

Proof. Necessity Part: Since $v \in \{0,1\}^{\binom{d}{2}}$ is a vertex of $\mathcal{T}_d$, thus $v \in \mathcal{C}_d$. We prove by contradiction. Suppose $v \in \mathcal{C}_d \backslash \mathcal{C}_d^o$, then there exists $1 \leq i, j, k, l \leq d$ such that $v_{ij} = v_{kl} = 1$ and $v_{ik} = v_{il} = v_{jk} = v_{jl} = 0$. This contradicts with what we have shown in Remark 7.

Sufficiency Part: Let $\{A_1, \ldots, A_k\}$ be a partition of $[d]$ with $\sum_{i=1}^{k} I_{|A_i|>1} \leq 1$. We first note that if $\sum_{i=1}^{k} I_{|A_i|>1} = 0$, then $k = d$. Additionally, A_i, for $1 \leq i \leq d$, are all singletons (*i.e.* sets with exactly one element) therefore the clique partition point of the partition $\{\{1\}, \ldots, \{d\}\}$ is the origin. The origin is a feasible point since we can set the parameters with respect to the Family BB1 to be $\lambda_i = \theta_i\delta_i = 0$, for $i = 1, \ldots, d$, respectively. Now we consider $\sum_{i=1}^{k} I_{|A_i|>1} = 1$. Without loss of generality, we set $|A_1| = d'$ for $1 < d' < d$, then the partition would be $\{A_1, \ldots, A_{d-d'+1}\}$, where $A_2, \ldots, A_{d-d'+1}$ are singletons. The clique partition point with respect to the partition is feasible by setting the parameters to be $\lambda_i = \theta_i\delta_i = \infty$, for $i = 1, \ldots, d'$ and $\lambda_i = \theta_i\delta_i = 0$, for $i = d'+1, \ldots, d$, respectively. □

Remark 8. It is noteworthy that the points of the set

$$\mathcal{C}_d^b \cap \mathcal{C}_d^o = \left\{ \pi(\{A_1, A_2\}) : \substack{\{A_1,A_2\} \text{ bipartition of } [d], \\ |A_1|\leq 1 \text{ or } |A_2|\leq 1} \right\} \tag{11}$$

can be generated from both the t copula and the one-factor copula with the linking copula being the Family BB1.

Now we consider the function $\phi(\cdot,\cdot)$ defined by

$$\phi(x, w) := (1 + w^x)^{-2/x-2} = e^{-(2/x+2)\ln(1+w^x)}, \quad x, w > 0.$$

We note that $\Psi(x,y) = \int_0^\infty (\phi(x,w)\phi(y,w))^{1/2} dw$ and $\Psi(x,x) = \int_0^\infty \phi(x,w)dw$, and the partial derivative of $\phi(x,w)$ with respect to x is given by

$$\frac{\partial \phi(x,w)}{\partial x} = (1+w^x)^{-2/x-2} \left(2x^{-2} \ln(1+w^x) - (2/x+2) \frac{w^x \ln w}{1+w^x} \right). \quad (12)$$

Lemma 8. *$\Psi(x,x)$ is strictly increasing continuous on $(0,\infty)$.*

Proof. By the same argument as Lemma 5, we need to prove

$$\int_0^\infty (1+w^x)^{-2x^{-1}-2} \left(2x^{-2} \ln(1+w^x) - \left(2x^{-1}+2\right) \frac{w^x \ln w}{1+w^x} \right) dw > 0.$$

Let $w^x = z$, then we need to prove

$$\int_0^1 2x^{-3} (1+z)^{-2/x-2} z^{1/x} \left((z+1/z) \ln(1+z) - z \ln z - (x+1) \frac{1-z}{1+z} \ln z \right) dz.$$

Since $(z+1/z)\ln(1+z) - z\ln z - (x+1)\frac{1-z}{1+z}\ln z > 0$, for $z \in [0,1]$ and $x > 0$, we conclude that $\partial \int_0^\infty \phi(x,w)dw / \partial x > 0$. □

We list some properties for $\Psi(x,y)$ in the following Proposition which we need for presenting further results.

Proposition 2. We have the following properties for $\Psi(x,y)$:

i. $\Psi(x,y) = \Psi(y,x)$, for any $x, y > 0$. Hence, the function is symmetric with respect to the line $y = x$.
ii. For any $a \in (0,1)$, there exists $x, y > 0$ such that $\Psi(x,y) = a$.
iii. For any $a \in (0,1)$ with $\Psi(x,y) = a$, there exists $l > 0$ such that $x, y \in [l, \infty)$.
iv. If $\Psi(x,y) = \Psi(x',y') = a$, for $a \in (0,1)$, then $I_{x \geq x'} + I_{y > y'} = 1$.

Proof. *i.* is trivial. *ii.*: Let $x = y$, so that Lemma 8 and the intermediate value theorem yields the above result. *iii.*: From Remark 6, we know that $a = \Psi(x,y) \leq \Psi_1(x,\infty)$. By the continuity of $\Psi_1(\cdot,\infty)$, let $l > 0$ be such that $\Psi_1(l,\infty) = a$, then the result followed by *ii.* and Lemma 7. *iv.*: If $x \geq x'$ (resp. $x < x'$), then by the monotonicity of $\Psi(x,\cdot)$, we have $y \leq y'$ (resp. $y > y'$). □

Now, we find the exact volume of the TDMs generated by the one-factor copula with the linking copula being the Family BB1 on $d = 3$. We start with one important property of its 3-dimensional tail dependence coefficients in the following proposition.

Proposition 3. If $\chi_{12}, \chi_{13}, \chi_{23} \in (0, 1]$, then

$$\begin{cases} \chi_{12} = \Psi(x, y) \\ \chi_{13} = \Psi(x, z) \\ \chi_{23} = \Psi(y, z) \end{cases}$$

can have at most one solution. Moreover, if (x, y, z) is the solution with $\chi_{12} \leq \chi_{13} \leq \chi_{23}$, then $x \leq y \leq z$.

Proof. If we suppose (x, y, z) and (x', y', z') are both the aforementioned solutions of the system of equations, then without loss of generality, we assume $x \geq x'$. By Proposition 2 *iv.*, we know that $y \leq y'$ and $z \leq z'$ from the upper two equations, and the third equation shows that $y = y'$ and $z = z'$. Hence, $x = x'$, then the two solutions are identical.
Now, suppose (x, y, z) is the solution of the system of equations with $\chi_{12} \leq \chi_{13} \leq \chi_{23}$, then by the monotonicity of $\Psi(x, \cdot)$ and the first two equations, we have $y \leq z$. By the same reasoning, we have $x \leq y$ from the last two equations. Thus, $x \leq y \leq z$. $\square$

In the following, we let $\mathcal{O}_d$, for $d \geq 3$, denote the sets of TDMs generated by the one factor copula with the linking copula being the Family BB1.

Proposition 4. $\mathcal{O}_d$ has volume if and only if $d = 3$.

Proof. Let $d = 3$ and ϕ be the mapping from the parameters of the one factor copula to $\mathcal{O}_3$. Then it can be shown as

$$\begin{aligned} \phi : \mathbb{R}_+^3 &\longrightarrow (0, 1)^3 \\ (x, y, z) &\longmapsto (\Psi(x, y), \Psi(x, z), \Psi(y, z)). \end{aligned}$$

We first proof that ϕ is homeomorphism by showing the following 3 properties: *i.* ϕ is a one-to-one function. We prove by contradiction. If there exist two different points (x_1, y_1, z_1) and (x_2, y_2, z_2) such that

$$(\Psi(x_1, y_1), \Psi(x_1, z_1), \Psi(y_1, z_1)) = (\Psi(x_2, y_2), \Psi(x_2, z_2), \Psi(y_2, z_2)),$$

without loss of generality, we could assume that $\Psi(x_1, y_1) \leq \Psi(x_1, z_1) \leq \Psi(y_1, z_1)$, then Proposition 3 yields $x_1 \leq y_1 \leq z_1$ and $x_2 \leq y_2 \leq z_2$. If $x_1 < x_2$, then from Proposition 2. *iv.*, we have $y_1 > y_2$ and $z_1 > z_2$. But $y_1 > y_2$ implies $z_1 < z_2$ which yields a contradiction. Hence $x_1 \geq x_2$. Similarly, we could prove that $x_1 \leq x_2$. Therefore, $x_1 = x_2$. By symmetry, we conclude that $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ which is a contradiction. *ii.* ϕ is continuous. It is clear that $\Psi(\cdot, \cdot)$ is continuous. Then ϕ is a 3-component function in which the coordinates are all continuous, thus ϕ is continuous. *iii.* ϕ^{-1} is continuous. Since Ψ is strictly increasing function, then Ψ^{-1} is continuous. Hence, ϕ^{-1} is continuous. Therefore, $\mathcal{O}_3$ is open by the homeomorphism application and the fact that $\mathbb{R}^3_+$ is open. Hence, $\mathcal{O}_3$ has volume.

For $d \geq 4$,

$$\phi : \mathbb{R}^d_+ \longrightarrow (0, 1)^{\binom{d}{2}}$$

is a mapping from lower dimensional space to higher. Hence, $\mathcal{O}_d$, for $d \geq 4$, does not have volume. □

Before digging into properties for high dimensions, we study the richness on dimension 3.

Remark 9. We note that $\mathcal{O}_3$ is not convex. If we consider two points, $(1, 0, 0)$ and $(0, 1, 0)$, which are both in $\mathcal{O}_3$, but from Remark 6, the middle point $(1/2, 1/2, 0)$ is not in $\mathcal{O}_3$.

From Proposition 3, one can see that $\mathcal{O}_3$ is symmetric by the indices. Without loss of generality, we can focus on the subset of the TDMs by assuming $\chi_{12} \geq \chi_{13} \geq \chi_{23}$. We denote the ordering area by $\widetilde{\mathcal{O}_3}$, which is the cone with the vertices $(0, 0, 0)$, $(1, 0, 0)$ and $(1, 1, 1)$, and which the 1/6 area of $\mathcal{O}_3$ (see figure 1).

Remark 10. We remark that $\widetilde{\mathcal{O}_3}$ is not convex. If we consider two points, $(1, 0, 0)$ and $(1/2, 1/2, 1/3)$, which are both in $\widetilde{\mathcal{O}_3}$, where the middle point $(3/4, 1/4, 1/6)$ is not in $\widetilde{\mathcal{O}_3}$. Additionally, the non-convexity of $\widetilde{\mathcal{O}_3}$ implies the non-convexity of $\mathcal{O}_3$ shown in Remark 9.

Proposition 5. The volume of $\mathcal{O}_3$ is 0.128820.

Proof. We focus on the ordering area $\widetilde{\mathcal{O}_3}$. Thus we need to find the volume of $(\Psi(x, y), \Psi(x, z), \Psi(y, z))$, where $0 \leq z \leq y \leq x$. By the monotonicity of the coordinates of $\widetilde{\mathcal{O}_3}$ presented in Lemma 5 and Proposition 3, for any fixed

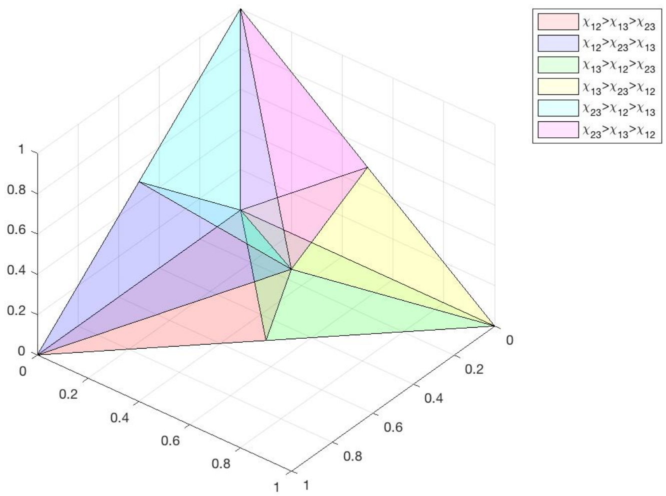

Figure 1: Ordering area of $\mathcal{O}_3$

$\Psi(x,y) := a$ and $\Psi(x,z) := b$, for $0 \le b \le a \le 1$, the lower bound for y and z are $\Psi^{-1}(\infty, a)$ and $\Psi^{-1}(\infty, b)$, respectively. Hence, the lower bound for $\Psi(y,z)$ is given by

$$\Psi\left(\Psi^{-1}(\infty, a), \Psi^{-1}(\infty, b)\right).$$

The surface of the above lower bound is shown in figure 2, while the area above the surface is that of $\widetilde{\mathcal{O}_3}$. We note that the volume of the surface and the $x-y$ plane is given by

$$\int_0^1 \int_0^a \Psi\left(\Psi^{-1}(\infty, a), \Psi^{-1}(\infty, b)\right) db\, da \approx 0.145197.$$

Thus the volume of $\widetilde{\mathcal{O}_3}$ is $(1/6 - 0.145197) \approx 0.021470$, and the result yields by times 6. □

Remark 11. The volume of $\mathcal{T}_3$ is 0.5, thus the volume of $\mathcal{O}_3$ is very close to the 1/4 volume, which equals to 0.125, of $\mathcal{T}_3$.

Remark 12. If we choose the linking copula from other BB families listed in Joe [14], then the TDMs generated by the one-factor copula would yield a different volume on $d = 3$ of the one-factor copula being the selected linking

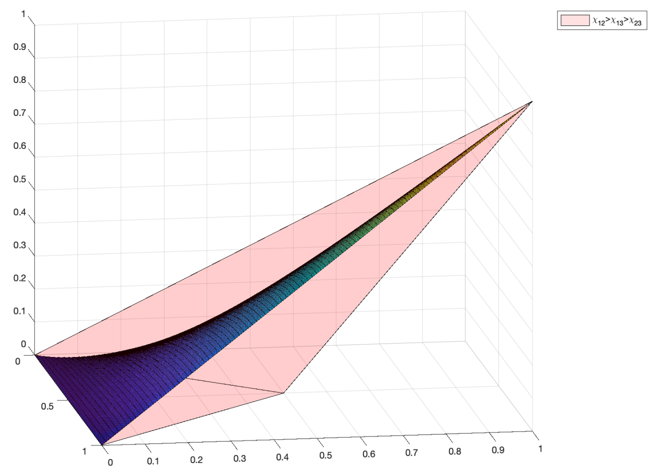

Figure 2: The surface of the lower bound of $\Psi(y, z)$

copulas. We list the properties of the related functions and the volume of the one-factor copula with the linking copula being other two BB families (BB4 and BB7), which are also suggested from the section 8.6 of Joe [15], in Table .3 in the Appendix.

Now, we consider $\mathcal{O}_4$. We note that from Proposition 4, $\mathcal{O}_d$, for $d \geq 4$, are all sets of isolated points. For any tail dependence coefficient of $\mathcal{O}_4$, $\bar{\chi} := (\chi_{1,2}, \ldots, \chi_{3,4})$, we order its coordinates by $\chi_{i_1,j_1} \leq \cdots \leq \chi_{i_6,j_6}$ with $(i_m, j_m) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, for $m = 1, \ldots, 6$. We define the ordering numbers n_u, for $u = 1,2,3,4$, with respect to $\bar{\chi}$ by

$$n_u(\bar{\chi}) := \sum_{m=1}^{6} 2^m I_{(i_m = u \text{ or } j_m = u)}, \quad \text{for } u = 1,2,3,4.$$

Among all 720 kinds of ordering partitions of $\mathcal{O}_4$, we consider the subset which has the ordering property $n_1 < n_2 < n_3 < n_4$, denoting by $\widehat{\mathcal{O}}_4$. We note that $\widehat{\mathcal{O}}_4$ has 30 partitions (see table 4 in appendix). For each partition, we use `polymake` to calculate and list its volume in the last column of the table. By using Proposition 3, it is easy to see that $\widetilde{\mathcal{O}}_4$ is contained in only two parts of $\widehat{\mathcal{O}}_4$, which are listed in the following Proposition without proof.

Proposition 6. Among all 30 partitions of $\widehat{\mathcal{O}}_4$, only two partitions contain the isolated points of $\widetilde{\mathcal{O}}_4$:

$$
\begin{aligned}
1: &\quad \chi_{12} < \chi_{13} < \chi_{23} < \chi_{14} < \chi_{24} < \chi_{34};\\
7: &\quad \chi_{12} < \chi_{13} < \chi_{14} < \chi_{23} < \chi_{24} < \chi_{34}.
\end{aligned}
$$

It is interesting to point out that the partitions 1 and 7 are adjoined to each other with a common facet of dimension 5. For all 30 partitions of $\widetilde{\mathcal{O}_4}$, we draw the adjacency structure in figure .3 in the Appendix. The polytopes contained in blocks or connected with lines denote that they have a common facet of dimension 5.

Remark 13. For any point in $\widehat{\mathcal{O}}_4$, one could find the lower bound of the euclidean distance to $\widetilde{\mathcal{O}}_4$ by projecting it to part 1 or 7 of $\widetilde{\mathcal{O}}_4$. For instance, the lower bound from the vertex $(1, 0, 0, 0, 0, 1) \in \widehat{\mathcal{O}}_4$ to part 1 and 7 is $\sqrt{0.8}$.

4. Conclusion and Discussion

In this study, our focus was on developing the geometric properties for the TDMs generated by some commonly used copula families. We started with the t copula and found the set of all $\{0, 1\}$-valued feasible vertices for any dimension which we presented as a subset of the clique partition points. Then we reported the approximate volume of the TDMs generated by the t copula for $d \leq 6$.

In the cases of the one-factor copula family, we focused on the linking copula being the Family BB1. We showed the set of the $\{0, 1\}$-valued feasible vertices. We calculated the exact volume on $d = 3$ of the set of TDMs generated by the one-factor copula with the linking copula being the Family BB1, and we reported the volume of the linking copulas as originating from other BB families as well. For practical interests, we showed some geometric properties of the TDMs on $d = 4$.

In future, it would be interesting to construct a copula family that cover all the TDMs on any dimension. By choosing different linking copulas, it might be interesting to find the coverage of p-factor copulas, where $p \geq 2$.

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Table .3: The volume of the one-factor copula family with the linking copula being the BB4 and BB7 families

Family	BB4	BB7
$C(u,v;\theta_i,\delta_i)$, $i=1,2,3.$	$(u^{-\theta_i}+v^{-\theta_i}-1-[(u^{-\theta_i}-1)^{-\delta_i}$ $+(v^{-\theta_i}-1)^{-\delta_i}]^{-1/\delta_i})^{-1/\theta_i}$, $\theta_i\geq 0,\delta_i>0, i=1,2,3.$	$1-(1-[(1-\bar{u}_i^{\theta})^{-\delta_i}+(1-\bar{v}_i^{\theta})^{-\delta_i}-1]^{-1/\delta_i})^{1/\theta_i}$ $\theta_i\geq 1,\delta_i>0,\ i=1,2,3,$ $\bar{u}=1-u,\ \bar{v}=1-v.$
$\eta(s;\theta_i,\delta_i)$, $i=1,2,3.$	—	$1-[1-(1+s)^{-1/\delta_i}]^{1/\theta_i}$, $\theta_i\geq 1,\delta_i>0,\ i=1,2,3.$
$\lim_{u\downarrow 0} C_{i\|0}(u\|uw)$, $i=1,2,3.$	$[1+w^{\theta_i}-(1+w^{-\theta_i\delta_i})^{-1/\delta_i}]^{-1/\theta_i-1}$ $\cdot[1-(1+w^{-\theta_i\delta_i})],\quad i=1,2,3.$	$(1+w^{\delta_i})^{-1/\delta_i-1},\quad i=1,2,3.$
χ_{ij}, $1\leq i<j\leq 3.$	$\int_0^\infty[1+w^{\theta_i}-(1+w^{-\theta_i\delta_i})^{-1/\delta_i}]^{-1/\theta_i-1}$ $\cdot[1-(1+w^{-\theta_i\delta_i})]$ $\cdot[1+w^{\theta_j}-(1+w^{-\theta_j\delta_j})^{-1/\delta_j}]^{-1/\theta_j-1}$ $\cdot[1-(1+w^{-\theta_j\delta_j})]dw,\quad 1\leq i<j\leq 3$	$\int_0^\infty \left(1+w^{\delta_i}\right)^{-1/\delta_i-1}\left(1+w^{\delta_j}\right)^{-1/\delta_j-1}dw$, $1\leq i<j\leq 3.$
Approximate Volume	0.167	0.129

Number	Ascending Order of the Indices						Volume
1	12	13	14	23	24	34	31
2	12	13	14	23	34	24	24
3	12	13	14	24	23	34	27
4	12	13	14	34	23	24	22
5	12	13	14	24	34	23	17
6	12	13	14	34	24	23	17
7	12	13	23	14	24	34	72
8	12	13	23	14	34	24	54
9	12	13	24	14	23	34	32
10	12	13	34	14	23	24	28
11	12	13	24	14	34	23	18
12	12	13	34	14	24	23	18
13	12	13	23	24	14	34	81
14	12	13	23	34	14	24	81
15	12	13	24	23	14	34	43
16	12	13	34	23	14	24	47
17	12	13	24	34	14	23	18
18	12	13	34	24	14	23	18
19	12	13	23	24	34	14	56
20	12	13	23	34	24	14	76
21	12	13	24	23	34	14	28
22	12	13	34	23	24	14	41
23	12	13	24	34	23	14	24
24	12	13	34	24	23	14	27
25	12	34	13	14	23	24	48
26	12	34	13	14	24	23	36
27	12	34	13	23	14	24	72
28	12	34	13	24	14	23	36
29	12	34	13	23	24	14	63
30	12	34	13	24	23	14	45

Table .4: 30 partitions of $\widetilde{\mathcal{O}_4}$. The volume column report the real volume times 207360.

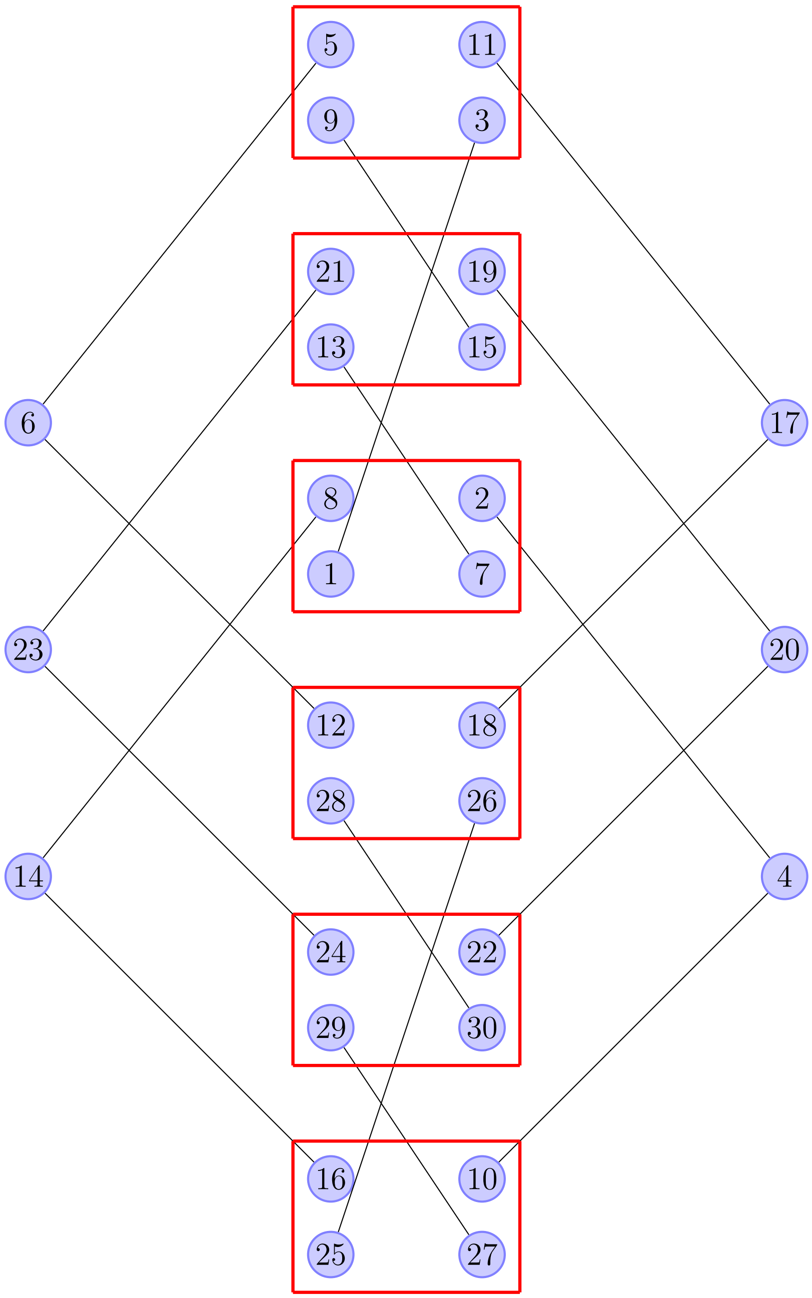

Figure .3: Construction of 30 parts of $\widetilde{\mathcal{O}_4}$. The polytopes in blocks or connected with lines are meaning that they have common facet of dimension 5. $\widehat{\mathcal{O}_4}$ contained in part 1 and 7.