A Note on Non-negative Continuous-time Processes

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Summary. Recently, there are much works on developing models suitable for analyzing the volatility of a continuous-time process. One general approach is to define a volatility process as the convolution of a kernel with a non-decreasing Lévy process, which is non-negative if the kernel is non-negative. Within the framework of Continuous-time Auto-Regressive Moving-Average (CARMA) processes, we derive a necessary and sufficient condition for the kernel to be non-negative. This condition is in terms of the Laplace transform of the CARMA kernel which has a simple form. We discuss some useful consequences of this result and delineate the parametric region of stationarity and non-negative kernel for some lower-order CARMA models.

Keywords: completely monotone; continuous-time ARMA process; Lévy process; Laplace transform; stochastic volatility

1. Introduction

Recently, there are much works on developing models suitable for analyzing the volatility of a continuous-time process, see Andersen and Lund (1997), Comte and Renault (1998) and Klüppelberg *et al.* (2004). Barndorff-Nielsen and Shephard (2001) considered a class of continuous-time stochastic volatility models for financial assets where the volatility processes are defined as solutions to Ornstein-Uhlenbeck (OU) processes driven by non-decreasing

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Lévy processes. The stationary OU process has the following moving-average representation:

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-u)} dL(u), \quad \lambda > 0, \tag{1}$$

where the kernel $g(t)$ defined by the formula $\exp(-\lambda t) I_{[0,\infty)}(t)$ is clearly non-negative. The driving Lévy process L is non-decreasing, so the process X defined by equation (1) is non-negative, making it applicable to volatility modeling. The autocorrelation function of X equals $\rho(h) = \exp(-\lambda h)$, which is monotonic decreasing. In practice, the autocorrelation function of the volatility process need not be monotone. Brockwell and Marquardt (2003) considered a class of Lévy-driven continuous-time ARMA (CARMA) processes of higher order and constructed a non-negative CAR(3) process that has a non-monotone autocorrelation function. Furthermore, they introduced a class of fractionally integrated Lévy-driven CARMA processes that are long memory extensions of the Lévy-driven CARMA processes.

For volatility modeling, the continuous-time process must be non-negative. An open, important problem with Brockwell and Marquardt's (2003) model concerns the necessary and sufficient condition for a general CARMA process driven by Lévy noise to admit a non-negative kernel. The purpose of this paper is to cast some light on this problem. We derive the Laplace transform of the kernel of a general CARMA process, which has a simple form. We show that the kernel is non-negative if and only if its Laplace transform is completely monotone, see Theorem 2 below. Based on this characterization, we give some more readily verifiable necessary condition for the kernel to be non-negative, as well some sufficient conditions for a non-negative kernel. This paper is organized as follows. In section 2, we briefly review the Lévy-driven CARMA processes of Brockwell and Marquardt (2003). The main results are stated in section 3. We characterize the parametric region of stationarity and non-negative kernel for some lower-order CARMA models in sections 3 and 4. These characterizations are pertinent for general volatility modeling with a possibly non-monotone autocorrelation function for the volatility process. All proofs are collected in an appendix.

2. CARMA Processes

We now recall the Lévy-driven CARMA(p, q) process of Brockwell (2000, 2001) and Brockwell and Marquardt (2003). The Lévy process is defined in terms of infinitely divisible distributions. Let $\phi(u)$ be the characteristic function of a distribution. We say that the distribution is infinitely divisible if, for every positive integer n, $\phi(u)$ is the nth power of some characteristic function. For every infinitely divisible distribution, we can define a stochastic process $\{X_t, t \geq 0\}$, called a Lévy process, such that it starts at zero and has independent and stationary increments with $(\phi(u))^t$ as the characteristic function of $X_{t+s} - X_s$, for any $s, t \geq 0$. For a detailed description of Lévy processes, see Protter (1991), Bertoin (1996) and Sato (1999). Heuristically, a Lévy-driven CARMA(p, q) process $\{Y_t\}$ is defined as some functional of the solution of a p-th order stochastic differential equation with suitable initial condition and driven by a Lévy process and its derivatives up to and including order $0 \leq q < p$. Specifically, for $t \geq 0$,

$$Y_t^{(p)} - \alpha_p Y_t^{(p-1)} - \cdots - \alpha_1 Y_t - \alpha_0 = \sigma\{L_t^{(1)} + \beta_1 L_t^{(2)} + \cdots + \beta_q L_t^{(q+1)}\}, \tag{2}$$

where $\{L_t, t \geq 0\}$ is a Lévy process with $EL_1^2 = 1$; the superscript $^{(j)}$ denotes j-fold differentiation with respect to t, i.e., $dY_t^{(j-1)} = Y_t^{(j)} dt, j = 1, ..., p-1$. We assume that $\sigma > 0$ and $\beta_q \neq 0$.

Equation (2) can be equivalently cast in terms of the *observation* and *state* equations (see Brockwell, 2001):

$$\begin{aligned} Y_t &= \beta' X_t, \qquad t \geq 0, \\ dX_t &= (AX_t + \alpha_0 \delta_p) dt + \sigma \delta_p dL_t, \end{aligned} \tag{3}$$

where the superscript $'$ denotes taking transpose,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_p \end{bmatrix}, \quad X_t = \begin{bmatrix} X_t^{(0)} \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{bmatrix}, \quad \delta_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-2} \\ \beta_{p-1} \end{bmatrix},$$

and $\beta_j = 0$ for $j > q$.

The process $\{Y_t, t \geq 0\}$ is said to be a CARMA(p,q) process with parameter $(\theta, \sigma) = (\alpha_0, ..., \alpha_p, \beta_1, \cdots, \beta_q, \sigma)$ if $Y_t = \beta' X_t$, where X_t is the solution of (3) with the initial condition X_0. Linearity of (3) implies that its solution can be written as

$$X_t = e^{At} X_0 + \alpha_0 \int_0^t e^{A(t-u)} \delta_p du + \sigma \int_0^t e^{A(t-u)} \delta_p dL_u,$$

where $e^{At} = I_p + \sum_{n=1}^{\infty}\{(At)^n(n!)^{-1}\}$, and I_p is the identity matrix.

For a random initial X_0, the mean vector of $\{X_t\}$, denoted by μ_t, satisfies the equation:

$$\begin{aligned}\mu_t &= e^{At}\mu_0 + \alpha_0 \int_0^t e^{A(t-u)}\delta_p du \\ &= e^{At}\mu_0 + \frac{\alpha_0}{\alpha_1}(e^{At} - I)\delta_1,\end{aligned}$$

where $\delta_1 = [1, 0, \cdots, 0]'$. If μ_0 is chosen to be $-(\alpha_0/\alpha_1)\delta_1$, then μ_t becomes $-(\alpha_0/\alpha_1)\delta_1$, which is independent of t. We assume that X_0 is independent of $\{L_t, t \geq 0\}$. $\{X_t, t \geq 0\}$ is strictly stationary if and only if all the eigenvalues of A have negative real parts and the initial distribution of X_0 equals that of $\int_0^\infty e^{A(t-u)}\delta_p dL_u$. The stationary CARMA process defined over non-negative t can be extended so that it is a stationary process over all real t. Let $\{M_t, 0 \leq t < \infty\}$ be a second Lévy process, independent of L and with the same distribution, and then define the following extension of L:

$$L_t^* = L_t I_{[0,\infty)}(t) - M_{-t-} I_{(-\infty,0]}(t), \qquad -\infty < t < \infty.$$

Then, provided all the eigenvalues of A have negative real parts, the process $\{X_t\}$ defined by

$$X_t = \sigma \int_{-\infty}^t e^{A(t-u)}\delta_p dL_u^*$$

is the strictly stationary solution of (3) for $t \in (-\infty, \infty)$ with the corresponding CARMA process given by

$$\begin{aligned}Y_t &= \sigma \int_{-\infty}^t \beta' e^{A(t-u)}\delta_p dL_u^* \\ &= \sigma \int_{-\infty}^\infty g(t-u) dL_u^*, \qquad -\infty < t < \infty,\end{aligned}$$

where $g(t) = \beta' e^{At}\delta_p I_{[0,\infty)}(t)$. In the case when the eigenvalues $\lambda_1, ..., \lambda_p$ are distinct and have negative real parts, Brockwell and Marquardt (2003) showed that

$$g(u) = \sum_{r=1}^p \frac{\beta(\lambda_r)}{\alpha^{(1)}(\lambda_r)} e^{\lambda_r u} I_{(0,\infty)}(u),$$

and the autocovariance function equals

$$\gamma(h) = \mathrm{cov}(Y_{t+h}, Y_t) = \sigma^2 \sum_{r=1}^p \frac{\beta(\lambda_r)\beta(-\lambda_r)}{\alpha^{(1)}(\lambda_r)\alpha(-\lambda_r)} e^{\lambda_r |h|}, \qquad (4)$$

where $\alpha(z) = z^p - \alpha_p z^{p-1} - \cdots - \alpha_1$, $\alpha^{(1)}$ denotes its first derivative and $\beta(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_q z^q$. Recall that the characteristic equation of A, i.e., $\det(A - zI) = 0$, equals

$\alpha(z) = 0$. We assume that all roots of $\alpha(z) = 0$ and those of $\beta(z) = 0$ have negative real parts. The condition on the roots of $\alpha(z) = 0$ is necessary for the stationarity of the process whereas that on $\beta(z) = 0$ is akin to the invertibility condition for discrete-time processes.

3. Main Results

For $(p, q) = (1, 0)$, the kernel g is non-negative, and consequently, if the driving Lévy process L is non-decreasing, the process X will be non-negative as is necessary if it is to represent volatility. We shall characterize the non-negativity of the kernel for any CARMA(p, q) process with $0 \leq q < p$ in terms of its Laplace transform. For this purpose, we first recall the definition of the Laplace transform. Let f be a function defined on $[0, \infty)$. Its Laplace transform is the function φ defined for $\lambda \geq 0$ by the following equation:

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx.$$

We now derive the Laplace transform of $g(t)$, the kernel of a CARMA(p,q) process.

LEMMA 1. *The Laplace transform of $g(t)$ equals $\varphi(s) = \beta(s)/\alpha(s), s \geq 0$.*

The significance of the Laplace transform of the kernel lies in the well-known result that the non-negativity of the kernel is equivalent to the complete monotonicity of its Laplace transform. First, we recall the definition of complete monotonicity; see Feller (1971) for further discussion. A function φ on $(0, \infty)$ is said to be completely monotone if and only if it possesses derivatives $\varphi^{(n)}$ of all orders and

$$(-1)^n \varphi^{(n)}(\lambda) \geq 0, \qquad \lambda > 0, \qquad n = 0, 1, 2, ...$$

We can now state the main results.

THEOREM 2. *(a) For a stationary CARMA(p, q) process, the kernel g is non-negative if and only if its Laplace transform, $\beta(s)/\alpha(s), s > 0$, is completely monotone.*
(b) For a stationary CAR(p) process, if the real part of each pair of complex, conjugate eigenvalues of A (defined below (3)) is smaller than or equal to a uniquely associated real eigenvalue of A, then the kernel g is non-negative.
(c) Another sufficient condition for the kernel g of a stationary CAR(p) process to be non-negative is that all eigenvalues of A are real and negative.

(d) A necessary condition for the kernel g of a stationary CAR(p) process to be non-negative is that there exists a real eigenvalue of A not smaller than the real part of all other eigenvalues of A.

(e) For $1 \leq q < p$, a sufficient condition for the kernel g of a stationary CARMA(p, q) process to be non-negative is that all the roots of $\alpha(z) = 0$ and those of $\beta(z) = 0$ are negative real numbers and for $1 \leq k \leq q$,

$$\sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \lambda_i, \tag{5}$$

where $\lambda_p \leq \cdots \leq \lambda_1 < 0$ are the roots of $\alpha(z) = 0$ and $\gamma_q \leq \cdots \leq \gamma_1 < 0$ are the roots of $\beta(z) = 0$.

Remarks

1. The necessary condition stated in part (d) is not sufficient for the kernel to be non-negative. For example, let $i = \sqrt{-1}$ and consider a CAR(5) process with eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1.002+\pi i$, $\lambda_3 = -1.002-\pi i$, $\lambda_4 = -1.001+1.1\pi i$ and $\lambda_5 = -1.001-1.1\pi i$. Then it can checked that $g(2) = -1.036 \times 10^{-3} < 0$.

2. For a stationary CARMA(2, 1) process, a necessary and sufficient condition for the kernel to be non-negative is that the two roots of $\alpha(s) = 0$ are real (denoted by $\lambda_2 \leq \lambda_1 < 0$) and $1 + \lambda_1\beta_1 \geq 0$, where $-1/\beta_1$ is the root of $\beta(s) = 0$. The sufficiency part follows from part (e) of the above theorem and the fact that the condition given in (5) is equivalent to $1 + \lambda_1\beta_1 \geq 0$. We now verify the necessity part. To see that the roots of $\alpha(s) = 0$ must be real, suppose the roots are complex so that $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$, where $b \neq 0$, in which case

$$g(u) = \frac{e^{au}}{b} \sin(\eta + bu)\sqrt{(b\beta_1)^2 + (1 + \beta_1 a)^2},$$

where $\eta = \sin^{-1}(b\beta_1/\{(b\beta_1)^2 + (1+\beta_1 a)^2\}^{1/2})$. But then g cannot be non-negative, leading to a contradiction; hence the roots of $\alpha(s) = 0$ must be real. Consider the case that $\lambda_2 < \lambda_1 < 0$. Then, we have

$$\varphi(s) = \frac{1 + \beta_1 s}{s^2 - \alpha_2 s - \alpha_1} = \frac{a}{s - \lambda_1} + \frac{b}{s - \lambda_2},$$

where $a = (1 + \lambda_1\beta_1)/(\lambda_1 - \lambda_2)$ and $b = -(1 + \lambda_2\beta_1)/(\lambda_1 - \lambda_2)$, and

$$\frac{(-1)^k \varphi^{(k)}(s)}{k!} = \frac{a}{(s - \lambda_1)^{k+1}} + \frac{b}{(s - \lambda_2)^{k+1}}.$$

The necessity that $1 + \lambda_1\beta_1 \geq 0$ now follows from the preceding two equations. On the other hand, if $\lambda_2 = \lambda_1 < 0$, then

$$\frac{(-1)^k \varphi^{(k)}(s)}{k!} = \frac{1}{(s-\lambda_1)^{k+1}} + \frac{(k+1)(\beta_1\lambda_1 + 1)}{(s-\lambda_1)^{k+2}},$$

and so $1 + \lambda_1\beta_1 \geq 0$ is a necessary condition for the kernel to be non-negative.

4. Parametric Region of Stationarity and Non-negative Kernel

For stationary CAR(p) processes with $p \leq 4$, parts (b–d) of Theorem 2 lead to simple necessary and sufficient conditions for the kernel to be non-negative. Indeed, it is clear from part (b) of Theorem 2 that for $p \leq 2$, a necessary and sufficient condition for the kernel to be non-negative is that all roots of $\alpha(z) = 0$ are negative numbers. Thus, the region of stationarity and non-negative kernel is specified by the inequality $\alpha_1 < 0$, for $p = 1$, whereas for $p = 2$ it is delineated by the inequalities: $\alpha_1 < 0$, $\alpha_2 < 0$ and $\alpha_2^2 + 4\alpha_1 < 0$. We note that these constraints and (4) imply that the autocorrelation function must be monotonic decreasing for $p \leq 2$; hence within the CAR model framework, the third order is the least order for which the process may have a non-monotone autocorrelation function.

The third-order case is a bit more complex. For a CAR(3) process, $\alpha(z) = z^3 - \alpha_3 z^2 - \alpha_2 z - \alpha_1$. Let s_1, s_2 and s_3 be the three roots of $\alpha(z) = 0$. By 3.8.2 of Abramowitz and Stegun (1965) (see, also, `http://mathworld.wolfram.com/CubicEquation.html`), we have

$$\begin{aligned}
s_1 &= \frac{\alpha_3}{3} + (S+T), \\
s_2 &= \frac{\alpha_3}{3} - \frac{(S+T)}{2} + i\frac{\sqrt{3}(S-T)}{2}, \\
s_3 &= \frac{\alpha_3}{3} - \frac{(S+T)}{2} - i\frac{\sqrt{3}(S-T)}{2},
\end{aligned}$$

where $S = \{R + (Q^3 + R^2)^{1/2}\}^{1/3}$, $T = \{R - (Q^3 + R^2)^{1/2}\}^{1/3}$, $Q = -(3\alpha_2 + \alpha_3^2)/9$ and $R = (9\alpha_2\alpha_3 + 27\alpha_1 + 2\alpha_3^3)/27$.

Let $D = Q^3 + R^2$, whose value of D determines the number of real and complex roots of $\alpha(s) = 0$:

(i) If $D > 0$, then $\alpha(z) = 0$ has one real root and a pair of complex conjugate roots. In this case, S and T are real, so we need the inequality $\alpha_3/3 - (S+T)/2 < \alpha_3/3 + (S+T) <$

0 in order for the kernel to be non-negative and for the process to be stationary. Equivalently, the above inequality can be rewritten as $0 < S + T < -\alpha_3/3$.

(ii) If $D = 0$, then all roots of $\alpha(z) = 0$ are real and at least two are equal, and $s_1 = \alpha_3/3 + 2R^{1/3}$, $s_2 = s_3 = \alpha_3/3 - R^{1/3}$. Therefore, the inequality $\alpha_3/3 < R^{1/3} < -\alpha_3/6$ is needed for stationarity and a non-negative kernel.

(iii) If $D < 0$, then all roots of $\alpha(z) = 0$ are real and distinct. Define $\theta = \cos^{-1}(R/\sqrt{-Q^3})$. Then, the real roots of $\alpha(z) = 0$ are

$$
\begin{aligned}
s_1 &= 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) + \frac{\alpha_3}{3}, \\
s_2 &= 2\sqrt{-Q}\cos\left(\frac{\theta + 2\pi}{3}\right) + \frac{\alpha_3}{3}, \\
s_3 &= 2\sqrt{-Q}\cos\left(\frac{\theta + 4\pi}{3}\right) + \frac{\alpha_3}{3},
\end{aligned}
$$

all of which must be negative for stationarity and a non-negative kernel.

For $p = 4$, a necessary and sufficient condition for stationarity and a non-negative kernel is that the roots of $\alpha(z) = 0$ either (i) are all negative or (ii) have a pair of complex roots and two real, negative roots, with the largest negative root being larger than or equal to the real parts of the other three roots. The roots of a quartic equation admit a closed-form, algebraic solution, see Abramowitz and Stegun (1965, pp. 17-18); in principle, so does the parametric region of stationarity and non-negative kernel of the CAR(4) model. But its derivation is not pursued here as it appears rather complex and that the parametric region of interest can be more conveniently delineated numerically.

These results indicate that in using CARMA models for modeling volatility processes, the parameters are generally subject to rather complex constraints. It is an interesting future research problem to develop efficient constrained CARMA estimation schemes for modeling volatility processes; see Roberts *et al.* (2004) for Bayesian modeling for a CAR(1) volatility model. Another interesting research problem consists of studying practical procedures for verifying condition (a) of Theorem 2 for other higher order CARMA processes.

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Appendix 1.

Proof of Lemma 1

Note that

$$\begin{aligned}\varphi(s) &= \int_0^\infty e^{-su} g(u)du \\ &= \int_0^\infty \beta' e^{(A-sI)u}\delta_p du \\ &= -\beta'(A-sI)^{-1}\delta_p \\ &= \frac{\beta(s)}{\alpha(s)}.\end{aligned}$$

To see the last equality, let $c = (A - sI)^{-1}\delta_p$. Write $c = (c_1, \cdots, c_p)'$. Consequently, $(A - sI)c = \delta_p$, which amounts to the following system of equations.

$$\begin{aligned} c_2 - sc_1 &= 0 \\ c_3 - sc_2 &= 0 \\ &\vdots \\ c_p - sc_{p-1} &= 0 \\ \alpha_1 c_1 + \cdots + \alpha_p c_p - sc_p &= 1. \end{aligned} \tag{6}$$

Thus, $c_j = s^{j-1}c_1$ for $p \geq j \geq 1$. Upon substituting these equations into (6), we have $-\alpha(s)c_1 = 1$. Therefore, $c_j = -s^{j-1}/\alpha(s)$. In particular, $-\beta'(A - sI)^{-1}\delta_p = -\beta'c = \beta(s)/\alpha(s)$. This completes the proof of the lemma.

Proof of Theorem 2

(a) This follows from Theorem 1 on page 439 of Feller (1971).

(b) The cases when $p = 1$ and $p = 2$ are trivial because the egenvalues of A must then be negative real numbers; consequently the claim follows from (c) of Theorem 2. The case of $p = 3$ can be proved as follows. Let the eigenvalues be λ, β_1 and β_2, where $\beta_1 = a + bi$, $\beta_2 = \bar{\beta}_1 = a - bi$, $a \leq \lambda < 0$, $b > 0$. Then

$$\begin{aligned} g(u) &= \frac{e^{\lambda u}}{|\lambda - \beta_1|^2} + \frac{e^{\beta_1 u}(\bar{\beta}_1 - \lambda) - e^{\bar{\beta}_1 u}(\beta_1 - \lambda)}{|\beta_1 - \lambda|^2(\beta_1 - \bar{\beta}_1)} \\ &= \frac{e^{\lambda u}}{|\lambda - \beta_1|^2} + \frac{e^{au}\{e^{ibu}(\bar{\beta}_1 - \lambda) - e^{-ibu}(\beta_1 - \lambda)\}}{|\beta_1 - \lambda|^2(\beta_1 - \bar{\beta}_1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{\lambda u}}{|\lambda-\beta_1|^2} + \frac{2ie^{au}\{(a-\lambda)\sin(bu) - b\cos(bu)\}}{|\beta_1-\lambda|^2 2bi} \\
&= \frac{e^{au}}{|\lambda-\beta_1|^2}\left\{e^{(\lambda-a)u} + \frac{(a-\lambda)\sin(bu) - b\cos(bu)}{b}\right\}.
\end{aligned}
$$

Note that $e^x \geq 1+x$ for all real x, which implies that $be^{(\lambda-a)u} \geq b+(\lambda-a)ub \geq b\cos(bu) + (\lambda-a)\sin(bu)$; therefore, $g(u) \geq 0$, for all $u > 0$. Equivalently, for a stationary CAR(3) process, $\varphi(s)$ is complete monotone. For $p > 3$, the results follow from the fact that the product of two complete monotone functions is still complete monotone (see Criterion 1 on page 441 of Feller, 1971).

(c) This follows readily from (i) the factorization $\varphi(s) = \prod_i (s-\lambda_i)^{-1}$, (ii) any function $(s-\lambda)^{-1}$ is completely monotone for a negative λ and (iii) the aforementioned result of Feller (1971, p.441).

(d) We prove the necessary condition, first for the simple case that all eigenvalues of A are distinct.

Denote the p distinct eigenvalues by $\lambda_i, i = 1, \cdots, p$. Suppose that there exists a complex eigenvalue whose real part is larger than the real part of any other eigenvalue of A. Without loss of generality, let λ_1 be such a complex eigenvalue, and that $\lambda_2 = \bar{\lambda}_1$, the complex conjugate of λ_1. Because $|s-\lambda_i|^2 = s^2 - 2\text{Re}(\lambda_i)s + |\lambda_i|^2$, for all sufficiently large real s, $|s-\lambda_i| > |s-\lambda_1|$, where $\text{Re}(\cdot)$ denotes the real part of the complex number in parentheses. (Recall that stationarity of the process implies that all eigenvalues have negative real parts.) By partial fraction, we get

$$
\varphi(s) = \sum_{i=1}^{p} \{\alpha^{(1)}(\lambda_j)(s-\lambda_j)\}^{-1},
$$

and hence

$$
\frac{(-1)^n \varphi^{(n)}(s)}{n!} = \sum_{i=1}^{p} \{\alpha^{(1)}(\lambda_j)\}^{-1}(s-\lambda_j)^{-n-1}.
$$

The first two terms in the sum are dominating and for sufficiently large n, the sign of the sum is same as that of the sum of the first two terms. Denote the sum of the first two terms by h_n which equals $2\text{Re}(\alpha^{(1)}(\lambda_1)(s-\lambda_1)^{n+1})/|\alpha^{(1)}(\lambda_1)(s-\lambda_1)^{n+1}|^2$. We claim that there exists infinitely many n for which the numerator of h_n is negative and hence $\varphi(s)$ is not completely monotone. We now prove the preceding claim. Let $\alpha^{(1)}(\lambda_1) = A\exp(i\eta)$ and $s - \lambda_1 = B\exp(i\theta)$, so that $\alpha^{(1)}(\lambda_1)(s-\lambda_1)^{n+1} = AB^{n+1}\exp(i\{\eta + (n+1)\theta\})$. Note that θ

is not a multiple of 2π because λ_1 is complex with a negative real part, lest both $s - \lambda_1$ and s are real implying the contradiction that λ_1 is real. Below, let s be a fixed and sufficiently large positive real number such that $s - \lambda_1$ has a positive real part. Hence, θ can be chosen such that it is either strictly between $-\pi/2$ and 0 or strictly between 0 and $\pi/2$. Consider the case that θ is between 0 and $\pi/2$. There exists a positive integer K such that $K\theta$ is strictly between $\pi/2$ and π. Suppose that for all sufficiently large n, the numerator of h_n is positive. Consequently, $\eta + (n+1)\theta$ is strictly between 0 and $\pi/2$ or strictly between $-\pi/2$ and 0 (modulus 2π), for all large enough n. Suppose that $\eta + (n+1)\theta$ is strictly between 0 and $\pi/2$. Then, $\eta + (n+1+K)\theta$ must be strictly between $\pi/2$ and $3\pi/2$. Similarly, if $\eta + (n+1)\theta$ is strictly between $-\pi/2$ and 0, $\eta + (n+1-K)\theta$ must be strictly between $-3\pi/2$ and $-\pi/2$. Hence, we have a contradiction so that for all sufficiently large s, $\varphi^{(n)}(s)$ is negative for some sufficiently large n that may depend on s. The case when θ is between $-\pi/2$ and 0 can be proved similarly. Hence, the eigenvalue with the largest real part must be real. Such an eigenvalue has the smallest magnitude as the real part of all eigenvalues are negative. Therefore, the non-negativity of the kernel g implies that the eigenvalue of A that is of smallest magnitude must be real. The proof for the case that A has multiple eigenvalues can be proved similarly, and hence omitted.

(e) This follows from Theorem 1 of Ball (1994).

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