22S:105
Statistical Methods and Computing

Introduction to Probability

Lecture 9
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Why do we want to study probability

- So far we have studied *descriptive statistics*: methods of describing or summarizing a *sample*
- We want to move ahead to *inferential statistics*: methods for using the data in a sample to draw conclusions about the *population* from which the sample is drawn.
- Methods of inferential statistics are based on the question "How often would this method give a correct answer if I used it very, very many times?"
- The laws of probability relate to this question.

Parameters and statistics

- A **parameter** is a numeric quantity that describes a characteristic of a population.
  - We almost never can know the exact value of a parameter, because we would have to measure every member of the population.
  - Example: We would like to know the average percent body fat of all Chinese males aged 21 - 65 years.
  - We generally use Greek letters to refer to population parameters.
  - μ is the standard symbol for a population mean.

- A **statistic** is a numeric value that can be computed directly from sample data.
  - Example: we draw a sample of 10 Chinese males aged 21-65 years and measure the percent body fat of each one.
    - The sample mean $\bar{x}$ of the 10 data values is a statistic.
  - We do *not* need to use unknown parameters to compute a statistic.
  - We often use a statistic to *estimate* and unknown parameter.
  - But the exact value of a particular statistic will be different in different samples drawn from the same population.
    - **sampling variability**

Randomness

- Chance behavior is unpredictable in the short run but has a predictable pattern in the long run.
- Example: tossing a coin
  - The proportion of heads in a small number of coin tosses is very variable.
  - As more and more tosses are done, the proportion settles down. It gets close to 0.5 and stays there.

- French naturalist Count Buffon (1707-1788) tossed a coin 4040 times and got 2048 heads.
  - proportion heads: $\frac{2048}{4040} = 0.5069$
- While imprisoned by the Germans during World War II, South African mathematician John Kerrich tossed a coin 10,000 times and got 5067 heads.
  - proportion heads: $\frac{5067}{10{,}000} = 0.5067$
- In 1900 English statistician Karl Pearson tossed a coin 24,000 times and got 12,012 heads.
  - proportion heads: $\frac{12012}{24000} = 0.5005$
- American statistician Kate Cowles (19?? - 20??) tossed a coin 5 times and got 4 heads
  - proportion heads: $\frac{4}{5} = 0.8$
- She repeated the experiment and got 2 heads
  - proportion heads: $\frac{2}{5} = 0.4$

Randomness

An experiment or observation is called **random** if individual outcomes are uncertain but there is a regular distribution of outcomes in a large number of independent repetitions.

Examples:

- We flip a coin and record the outcome as a head or tail
- We draw an 18-year-old American male "at random" and follow up to find out whether he lives to be 65
- We draw an American child at random and record his/her position in birth order of children in the family
- A researcher feeds a baby rat a particular diet and records the rat's weight gain from birth to age 30 days

The **sample space S** is the set of all possible outcomes of a random experiment.

Examples:

- We flip a coin and record the outcome as a head or tail
- We draw an 18-year-old American male "at random" and follow up to find out whether he lives to be 65
- We draw an American child at random and record birth order
- A researcher feeds a baby rat a particular diet and records the rat's weight gain from birth to age 30 days

An **event** is any outcome or set of outcomes of a random experiment.

Example: At random, we draw a child born in the US and record his/her live birth order. We would observe one of the following events:

She is

- 1st child
- 2nd child
- 3rd child
- 4th child
- 5th child
- 6th or later

Or, we might lump certain outcomes together into a single event of interest.

- Child is "1st child" or "not 1st child"

Capital letters near the beginning of the alphabet often are used to denote events.

Example:

- A might represent the event that the child is a 1st child.
- B might represent the event that the child is not a first child.

The **probability** of an event is the proportion of times the outcome would occur in a very long series of repetitions under the same conditions.

- (This is the "long-run frequency" definition of probability.)
- coin tosses: the probability of getting a head is 0.5
- birth order of randomly drawn American child

Birth order	1st	2nd	3rd	4th	5th	6+
Probability	0.416	0.330	0.158	0.058	0.021	0.017

The probability that an event occurs is often denoted with the letter P.

- $P(A)$ is the probability of event A

More probability terminology

- The event (A does not occur) is called the *complement of A* and represented by A^c
  - If A is the event that the randomly drawn child is a first-born child, then what is A^c?
- Two events A and B that cannot occur simultaneously are *disjoint* or *mutually exclusive*
- The *union* of two events is the event that one or the other or both occur.
  - The union of events A and B is the event (A or B or both)
- The *intersection* of two events is the event that both occur.

- The intersection of events A and B is the event (A and B)

- Two events A and B are *independent* if the probability that one occurs does not change the probability that the other one occurs.
  - Example: Suppose one person tosses a penny and another person tosses a dime. The outcomes of the two tosses are independent. Each has a probability of $\frac{1}{2}$ of being a head. The outcome for one of the coins has no effect on the probabilities of the two possible outcomes for the other coin.
  - Example 2: What if the same person tossed the same coin twice?

- Example 3: I have a deck of cards. I draw a card at random. Without putting it back, I draw a second card at random.

  The event A is that the first card is a heart. The event B is that the second card is a heart. Are events A and B independent?

Probability models

- mathematical models for randomness!
- consist of two parts
  - a sample space S
  - a way of assigning probabilities to events

Probability rules

1. Any probability is a number between 0 and 1.

   If P(A) is the probability of any event A, then
   $$0 \leq P(A) \leq 1$$

2. All possible outcomes taken together must have probability 1.
   $$P(S) = 1$$
   - One of the possible outcomes has to happen!

3. The probability that an event does not occur is 1 minus the probability that the event does occur.
   - $P(A^c) = 1 - P(A)$
   - Example: If the probability that a randomly selected black American has type O blood is 0.49, what is the probability that he or she has some other blood type?

4. (Addition rule): If two events are mutually exclusive, then the probability that one or the other occurs is the sum of their individual probabilities.
   - If A and B are disjoint events, then
     $$P(A \;\; or \;\; B) = P(A) + P(B)$$
   - Example: For our child, we might wish to define an event as
     - "1st or 2nd child" = either "1st child" or "2nd child"
     - Since being a 1st child and being a 2nd child are mutually exclusive events then
       $$\begin{aligned} P(1st \;\; or \;\; 2nd) &= P(1st) + P(2nd) \\ &= 0.416 + 0.330 \\ &= 0.746 \end{aligned}$$

5. This rule can be extended to three or more mutually exclusive events.
   - If A, B, and C are all mutually exclusive then
     $$P(A \; or B \;\; or \;\; C) = P(A) + P(B) + P(C)$$
   - Example:
     $$\begin{aligned} P(1st, \;\; 2nd, \;\; or \;\; 34d) &= P(1st) + P(2nd) + P(3rd) \\ &= 0.416 + 0.330 + 0.158 \\ &= 0.904 \end{aligned}$$

     How else might we have computed P(1st, 2nd, or 3rd)?

6. (Multiplication rule for independent events): If two events A and B are independent,
   $$P(A \;\; and \;\; B) = P(A) \;\; P(B)$$
   - Example: Suppose I have two separate, complete decks of cards (52 cards in each).
     - I draw one card at random from the first deck. What is the probability that that card is a heart?
     - If I draw one card at random from the first deck and another card at random from the second deck, what is the probability that *both* cards are hearts?

Assigning probabilities when the sample space is finite

- Assign a probability to each individual outcome.
- These probabilities must all be numbers between 0 and 1, and they must sum to 1.
- Example: Our table of probabilities of the birth positions of American kids is a probability model.

Birth order	1st	2nd	3rd	4th	5th	6+
Probability	0.416	0.330	0.158	0.058	0.021	0.017